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On the weakly nonlinear three-dimensional instability of shear layers to pairs of oblique waves: the Stokes layer as a paradigm

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The nonlinear evolution of a pair of initially linear oblique waves in a high-Reynolds-number shear layer is studied. Attention is focused on times when disturbances of amplitude ϵ have $O(\epsilon^{\frac{1}{3}}R)$ growth rates, where R is the Reynolds number. The development of a pair of oblique waves is then controlled by nonlinear critical-layer effects (Goldstein & Choi 1989). Viscous effects are included by studying the distinguished scaling $\epsilon = O(R^{-1})$. When viscosity is not too large, solutions to the amplitude equation develop a finite-time singularity, indicating that an explosive growth can be induced by nonlinear effects; we suggest that such explosive growth is the precursor to certain of the bursts observed in experiments on Stokes layers and other shear layers. Increasing the importance of viscosity generally delays the occurrence of the finite-time singularity, and sufficiently large viscosity may lead to the disturbance decaying exponentially. For the special case when the streamwise and spanwise wavenumbers are equal, the solution can evolve into a periodic oscillation. A link between the unsteady critical-layer approach to high-Reynolds-number flow instability, and the wave/vortex approach of Hall & Smith (1991), is identified.

1 Introduction

The aim of this paper is to explain one of the mechanisms by which nonlinear effects can play a role in the development of three-dimensional disturbances in high-Reynolds-number shear layers. The analysis applies to any 'nearly parallel' shear layer which is inviscidly unstable. In particular it applies both to shear layers that slowly develop downstream, e.g. the free shear layer behind a splitter plate, and to shear layers that slowly evolve in time, e.g. a Stokes layer. Where appropriate we will fix ideas by applying the theory to Stokes layers, i.e. the flow generated above a sinusoidally oscillating plane wall.

1.1. *The stability of Stokes layers*

The Stokes layer is one of the simplest exact unsteady solutions of the Navier–Stokes equations. Its instability has been studied as a paradigm of the instability of unsteady (periodic) flows. Although the flow is unidirectional, a conventional normal-mode approach to the linear stability of the flow is not possible due to the unsteadiness of the basic state. Instead von Kerczek & Davis (1974) and Hall (1978) used Floquet theory to seek linear disturbances which grow over a complete period. However, they found that over a full period the flow was stable at all Reynolds number investigated. At the highest end of the range studied this included Reynolds numbers for which instabilities have been observed experimentally (e.g. Merkli & Thomann 1975).

This paradox has been partly resolved by Tromans (1977) and Cowley (1987) who argued that at high Reynolds numbers the rapid growth of small high-frequency disturbances over part of a period can lead to nonlinear effects preventing the linear decay over a whole period (see also Hall 1983). This idea was developed by Wu & Cowley (1993) (see also Wu 1991) for two-dimensional disturbances using the unsteady, or non-equilibrium, critical-layer approach of Hickernell (1984), Churilov & Shukhman (1988), Goldstein & Leib (1989) and others. In particular, following Goldstein & Leib (1989) they found that nonlinear interactions inside the *critical layers* could affect the evolution of disturbances sufficiently to cause the amplitude to 'blow-up' in a finite time. Unfortunately, although the most unstable linear disturbances in Stokes layers are two-dimensional, there are as yet no sufficiently well-controlled experiments to compare this theory with. Indeed to the best of our knowledge there are *no* experiments on Stokes layers with controlled two-dimensional disturbances. In the experiments that have been performed the disturbances are three-dimensional.

Of course the importance of three-dimensionality has long been realized in transition. For instance, three-dimensional perturbations are the most unstable disturbances in compressible supersonic shear layers. Further, in the case of the Tollmien–Schlichting instability of subsonic boundary layers, disturbances are predominantly two-dimensional only in carefully controlled experiments, e.g. Schubauer & Skramstad (1947), Nishioka, Iida & Ichikawa (1975). Even then two-dimensional perturbations dominate only in the early stages of transition, with three-dimensional disturbances growing to significance downstream, e.g. Klebanoff, Tidstrom & Sargent (1962), Kachanov & Levchenko (1984), Kachanov (1987), Saric & Thomas (1984).

Here we extend our analysis of instability and transition in Stokes layers to three-dimensional disturbances consisting of a pair of oblique waves. However, in order to maintain maximum generality we develop our theory for a general velocity profile, and then specialize to Stokes layers in the conclusions.

As in any three-dimensional nonlinear stability analysis a number of theoretical methods are potentially available. One of the most well-known approaches is a weakly nonlinear expansion about a critical Reynolds number. However, for a *non-parallel* shear layer such a critical Reynolds number is not defined (e.g. see Smith 1979), while for a Stokes layer, linear Floquet theory has yet to yield a finite critical Reynolds number. Instead we assume that the Reynolds number, R, of the flow is large. In the case of a Stokes layer this means that the oscillation frequency, ω, is much smaller than a typical $O(\omega R)$ frequency of the instability waves (while for a non-parallel shear layer it means that the viscous development length is much larger than the wavelength of an instability wave). Under such conditions, linear instability waves are quasi-steady and satisfy Rayleigh's equation (e.g. Tromans 1979).

In order to fix ideas, consider figure 1 which is a graph of the neutral curves of a Stokes layer plotted as parametric functions of time. This is for a flow generated by a flat plate at $y^* = 0$ with velocity $(U_0 \cos \omega t^*, 0, 0)$, where (x^*, y^*, z^*) and t^* are dimensional Cartesian coordinates and time respectively. The Reynolds number and

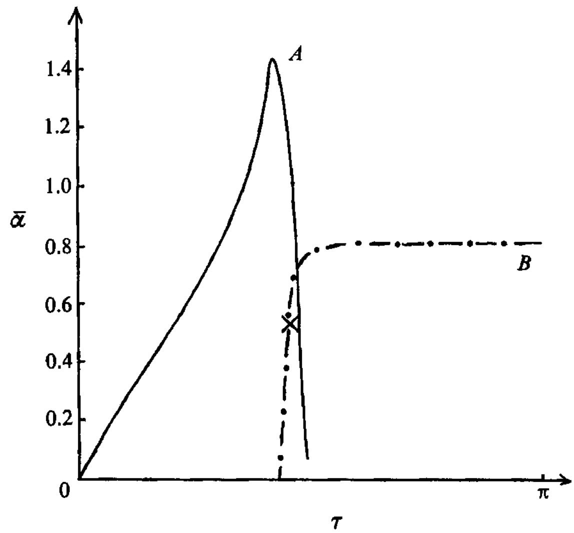

FIGURE 1. Sketch of the linear neutral curves of a Stokes layer plotted as parametric functions of time (from Cowley 1987); $\bar{\alpha} = (\alpha^2 + \beta^2)^{\frac{1}{2}}$ is the wavenumber and τ_0 is a point on one of the neutral curves. Unstable modes can be found for those wavenumbers and at those times beneath the two neutral curves A and B; $\times$ is a mode crossing point. In the analysis we concentrate on times close to $\tau = \tau_0 + \epsilon^{\frac{1}{3}}\tau_1$, where ϵ is the magnitude of the disturbance and τ_1 is an order-one number.

Stokes layer thickness are respectively

$$R = \left(\frac{2U_0^2}{\omega\nu}\right)^{\frac{1}{2}} \quad \text{and} \quad \delta^* = \left(\frac{2\nu}{\omega}\right)^{\frac{1}{2}},$$

and ν is the kinematic viscosity. The streamwise and spanwise wavenumbers are non-dimensionalized by δ^{*-1}, and are denoted by α and $\pm\beta$. At any time an 'infinite' number of Rayleigh modes exist, but the most rapidly growing modes can for the most part be found at those times and for those wavenumbers that lie beneath the solid curve A in figure 1; see Cowley (1987) for further details including the significance of the mode crossing point marked by $\times$.

In this paper we will not be concerned with how Rayleigh modes are excited in the shear layer, i.e. the receptivity problem. Instead we adopt the conventional assumption that as a result of background disturbances Rayleigh modes are introduced into the flow. For instance, suppose that a pair of oblique modes are introduced into a Stokes layer at a time and with a wavenumber given by the left-hand branch of one of the curves A or B. As the Stokes layer's mean profile slowly evolves this mode will begin to grow exponentially fast. A number of possibilities then arise:

(*a*) First suppose that although small, the initial amplitude is such that nonlinear effects come in almost immediately near the left-hand branch of either curve A or B. For definiteness we assume that a disturbance of wavenumber $\bar{\alpha} = (\alpha^2 + \beta^2)^{\frac{1}{2}}$ is introduced at the neutral time $\tau = \tau_0$, and that it then has an amplitude ϵ_0. At a time τ, where $\Delta\tau = \tau - \tau_0 \ll 1$, a simple linear quasi-steady theory gives the growth rate as $\dot{\sigma}_0 R\Delta\tau$ and the wave amplitude, ϵ, as

$$\epsilon \sim \epsilon_0 \exp\left(\tfrac{1}{2}\dot{\sigma}_0 R\Delta\tau^2\right),$$

where $\dot{\sigma}_0 \equiv \dot{\sigma}_0(\bar{\alpha})$ is known (Cowley 1987). Inviscid arguments similar to those of Goldstein & Choi (1989, henceforth referred to as GC) then show that nonlinear effects become important in an unsteady critical layer when the growth rate is $O(\epsilon^{\frac{1}{3}}R)$. Of course this scaling only applies if the flow becomes nonlinear while a critical layer exists, i.e. if $\Delta\tau \ll 1$, or equivalently if

$$2\log\epsilon_0^{-1} \ll \dot{\sigma}_0 R \ .$$

In addition, the initial amplitude ϵ_0 must not be too large, otherwise both nonlinear effects and viscous effects must be included in the critical layer. In particular, if

$$\log\epsilon_0^{-1} = O(R^{\frac{1}{3}}) \ ,$$

so that nonlinear effects come into play when $\epsilon = O(R^{-1})$ and $\Delta\tau = O(R^{-\frac{1}{3}})$, then the critical layer is unsteady and viscous. This is the scaling that will be studied here. Recently Hultgren (1992) has obtained excellent comparison between experiments and a theory based on unsteady non-equilibrium critical layers which is similar in spirit to that considered here. Hultgren's (1992) analysis suggests that theoretical results we obtain may have a wider range of validity than might first be thought in view of our specific choice of distinguished scaling. We delay until later a discussion of the 'very viscous' case that occurs when the flow becomes nonlinear for $\Delta\tau \ll R^{-\frac{1}{3}}$, i.e. when $\log\epsilon_0^{-1} \ll R^{\frac{1}{3}}$ (see §§4.2 and 4.4).

(*b*) Next suppose that the initial amplitude of the disturbance is sufficiently small that the modes evolve linearly until they approach the right-hand branch of curve A†, say at $\tau = \tau_0$ again. Then, as in (*a*), nonlinear effects can be accounted for by an unsteady, viscous, critical-layer analysis if $R^{-\frac{1}{2}} \ll |\Delta\tau| \ll 1$ (see §4.4 for a brief discussion of the very viscous case $\Delta\tau = O(R^{-\frac{1}{2}})$).

(*c*) The third possibility is that for disturbances of a given wavenumber, nonlinear effects become significant when the growth rate of the modes is $O(R)$, i.e. at a time far away from the neutral curves A and B. A fully nonlinear theory then seems necessary, and we do not consider this possibility further here. However, we note that asymptotic theories exist that modify the mean flow, so effectively changing the position of the neutral curves, e.g. the wave/vortex interaction theory of Hall & Smith (1991). The latter theory assumes that there is a weakly nonlinear, high-frequency, neutral wave (e.g. a Rayleigh neutral mode) whose evolution on a 'slow' timescale nonlinearly modifies the basic state by an order-one amount. However, in order for this interesting theory to be applicable, it is necessary for the flow to be able to evolve to the weakly nonlinear neutral wave in the first place. It turns out that our viscous generalization of GC's work is related to this question, and provides a link between two of the fashionable high-Reynolds-number stability theories (see §4.4).

A main concern of this paper will thus be with the nonlinear effects associated with unsteady, viscous *critical layers* of three-dimensional disturbances. Specifically we shall seek a possible mechanism that leads to the bursting phenomenon observed in experiments on Stokes layers (e.g. see Merkli & Thomann 1975; Hino, Sawamoto & Takasu 1976; Hino *et al.* 1983). We shall assume that the disturbance consists of a pair oblique waves because the 'quadratic interaction' of such waves produces a vortex flow, and we note that a significant vortex structure has been observed by Hino *et al.* (1983).

† The left-hand branch of such modes is on curve A for wavenumbers above point X of figure 1, and on curve B otherwise (Cowley 1987).

1.2. *Critical-layer theory*

There have been two main strands in the critical-layer theory of nonlinear flow stability (see the reviews by Stewartson 1981 and Maslowe 1986 for overviews of critical-layer theory and its applications). One strand has consisted of directly seeking nonlinear neutral-wave solutions, but without giving detailed consideration to whether the flow could evolve to these equilibrium states, e.g. Benney & Bergeron (1969), Smith & Bodonyi (1982*a,b*), Bodonyi, Smith & Gajjar (1983), Gajjar & Cole (1989) and Gajjar (1990). In the second strand, the evolution of a linear wave is followed into the nonlinear regime by introducing a slow time (temporal instability) or equivalently a 'slow' lengthscale (spatial instability), e.g. Churilov & Shukhman (1987*a*, 1988), Goldstein, Durbin & Leib (1987). We take the second approach here, and as is appropriate for Stokes layers, a *temporal* instability viewpoint is adopted. However, there are close analogies between the temporal instability of Stokes layers and the *spatial* instability of steady shear layers and (compressible) boundary layers (e.g. GC). This allows us both to draw on previous work and to extend our analysis in a straightforward manner to non-parallel shear layers (e.g. see Appendix A).

The instability modes we consider have a wavelength comparable with the thickness of the shear layer. In the case of *two*-dimensional modes, the scalings and nature of the nonlinear analysis depend crucially on whether the neutral modes have a regular or (logarithmically) singular critical layer. In the former case, which is so on most of the left-hand branch of curve A, a strongly nonlinear critical-layer analysis is necessary as in the work of Churilov & Shukhman (1987*a*), Goldstein, *et al.* (1987), Goldstein & Leib (1988), Leib & Goldstein (1989), Goldstein & Hultgren (1988) and Hultgren (1992) (for other examples of this type of critical layer see the work of Goldstein & Wundrow 1990 and Shukhman 1989). However, on the rest of curve A, and all of curve B, there are two or more singular critical layers. Close to these curves it is found that for a given growth rate, nonlinear effects are felt at smaller disturbance amplitudes than if the critical layers had been regular; as a result a *weakly* nonlinear analysis is possible. However, the amplitude equation is *not* a Stuart–Watson–Landau equation unless viscous effects are large; instead an integro-differential equation is recovered (Hickernell 1984; Churilov & Shukhman 1988; Shukhman 1991; Wu 1991; Wu & Cowley 1993).

In the case of *three*-dimensional disturbances, Benney (1961) observed that a nearly neutral Rayleigh mode has a pole singularity in the streamwise velocity at the critical layer. At first sight this suggests that a different scaling is necessary in order to follow the nonlinear evolution of three-dimensional modes. Nevertheless, in the case of a *single* oblique mode in a (compressible) shear layer, a Squire transformation enables the weakly nonlinear problem to be reduced to one in which the scalings are essentially those for a two-dimensional singular critical layer (Goldstein & Leib 1989; Leib 1991). However this is not possible in the case of a *pair* of oblique modes.

The nonlinear spatial evolution of a pair of oblique waves in a free shear layer has been studied by GC. They showed that because of the pole singularity, nonlinear effects must be included when the linear growth rate is $O(\epsilon^{\frac{1}{3}}R)$. Our temporal analysis of the development of a pair of oblique waves closely follows the spatial analysis of GC, although we additionally allow the critical layer(s) to occur *away* from inflexion points. For two-dimensional disturbances such a generalization leads to a completely different critical-layer structure. However in the three-dimensional case it does not affect the critical-layer dynamics if the spanwise wavenumber $\beta \gg \epsilon^{\frac{1}{3}}$ (Wu 1993*a*). Most importantly, in addition to this generalization we incorporate viscous effects so that the critical layers involved are unsteady and viscous in nature. This enables us to make a link between the unsteady-critical-layer approach to instability and the wave/vortex interaction approach (Hall & Smith 1991).

1.3. *The underlying scaling*

We are thus interested in the evolution of a pair of high-frequency oblique modes when nonlinear effects become important near a neutral curve, i.e. either soon after the modes become unstable or just before they stabilize. As explained in detail by GC, Wu (1991) and Wu & Cowley (1993), it is appropriate to concentrate on times close to

$$\tau = \tau_0 + \epsilon^{\frac{1}{3}}\tau_1 ,$$

for some suitable $\tau_1 = O(1)$, i.e. times at which the linear growth rate is $O(\epsilon^{\frac{1}{3}}R)$. Therefore we introduce the timescales

$$t_1 = \tfrac{1}{2}\epsilon^{\frac{1}{3}}R\tau , \tag{1.1}$$

and

$$t = R\tau \tag{1.2}$$

in order to account for the 'slow' nonlinear growth/decay of the disturbance, and the 'fast' carrier wave frequency of the disturbance, respectively.

The basic flow $\bar{U}$ evolves on the very slow timescale τ, and it turns out to be sufficient to express its profile at time τ as a Taylor series about the neutral time τ_0:

$$\bar{U}(y,\tau) = \bar{U}(y,\tau_0) + \epsilon^{\frac{1}{3}}\bar{U}_\tau(y,\tau_0)\tau_1 + \dots .$$

Hereafter all quantities associated with the basic flow will be evaluated at τ_0 unless otherwise stated.

In order to maintain maximum generality, we wish to retain viscous diffusion terms at leading order in the critical-layer equations. An elementary balance of the unsteady, u_τ, and viscous, $R^{-1}u_{yy}$, terms in the critical layer of width $\epsilon^{\frac{1}{3}}$ (see (2.30)), shows that we require

$$R^{-1} = \lambda\epsilon , \tag{1.3}$$

where the parameter λ reflects the relative importance of viscous to nonlinear effects (cf. the Haberman parameter). Throughout §§2 and 3, λ will be assumed to be of order one. The highly viscous case corresponding to λ being asymptotically large will be discussed in §§4.2 and 4.4.

The overall evolution of a three-dimensional disturbance is summarized in figure 2 for the case when the flow goes nonlinear near the right-hand branch of curve A. As illustrated, the disturbance is initially linear and grows exponentially until its growth rate decreases to $O(\epsilon^{\frac{1}{3}}R)$ when $\tau_1 = O(1)$. At this stage, nonlinear interactions inside the critical layers control the evolution. We wish to emphasize that there are four timescales illustrated in this figure:

(*a*) the very slow timescale, τ, over which the Stokes layer evolves;
(*b*) the slow timescale, τ_1, over which the growth rate evolves;
(*c*) the faster timescale, t_1, over which the disturbance grows;
(*d*) the fast timescale, t, over which the disturbance oscillates.

We note that although our analysis is based on being close to either the left- or right-hand branches of the neutral curve, it is straightforward to modify it to wavenumbers close to the apex of curve A in figure 1 (cf. Hickernell 1984).

The paper is organized as follows. In §2 we construct asymptotic perturbation

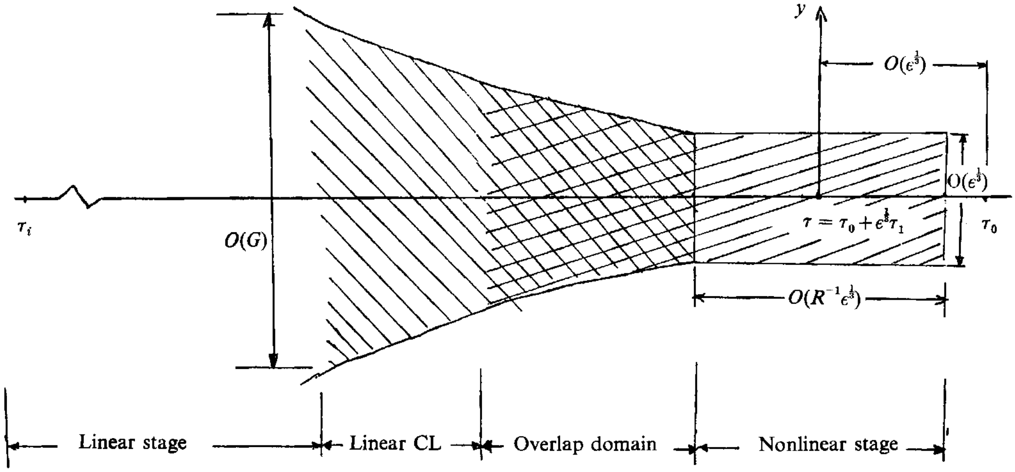

FIGURE 2. Evolution stages and critical-layer structures (for the case when nonlinearity becomes important near the right-hand branch of a neutral curve). The disturbance initially grows exponentially according to linear theory. As it approaches the neutral time τ_0, the growth rate becomes small, and linear critical layer(s) emerge. When the growth rate has decreased further to $O(\epsilon^{\frac{1}{3}}R)$, nonlinear interactions inside the critical layers control the evolution of the disturbance. The earlier linear, and the subsequent nonlinear, evolution stages match in the overlapping domain.

expansions in the 'outer' region away from the critical layers. The limiting forms of these solutions near the critical layers are then determined; as usual these contain unknown 'jumps' across the critical layers. A solvability condition is also deduced for an inhomogeneous Rayleigh equation. In § 3, we analyse the unsteady, viscous and weakly nonlinear flow within the critical layers. By matching the inner and outer solutions the unknown jumps are evaluated. Then by combining the solvability condition with these jumps, we derive the amplitude equation which is a main result of this paper. The amplitude equation is studied in § 4, both analytically and numerically. In particular, a finite-time singularity structure is identified as in GC, and confirmed by numerical solution. In addition, exponentially decaying solutions are found under certain conditions. The viscous limit is discussed and a link is established with the wave/vortex interaction work of Hall & Smith (1991). Finally, in § 5, we summarize our main results, and discuss the implications of this study. In Appendix A, as a demonstration of the general application of the present study, we deduce the amplitude equation for free shear layers by combining the present results with those of GC.

2 Outer expansion

We take the flow to be described by Cartesian coordinates $(x^*, y^*, z^*) = \delta^*(x, y, z)$, where x^* is parallel to the direction of oscillation of the plate, y^* is normal to the plate and z^* is the spanwise direction. We non-dimensionalize time with ω^{-1}, i.e. $\tau = \omega t^*$, and write the velocity as $U_0(U, V, W)$. The analysis applies to any inviscidly unstable almost parallel two-dimensional velocity profile $(\bar{U}, R^{-1}\bar{V}, 0)$. However, for purposes of illustration we will substitute at appropriate points the Stokes-layer solution for flow over an oscillating plate:

$$(\bar{U}, R^{-1}\bar{V}, 0) = (\cos(\tau - y)\mathrm{e}^{-y}, 0, 0)\ .$$

We denote the perturbed flow by

$$(U, V, W) = (\bar{U} + u, R^{-1}\bar{V} + v, w) .$$

2.1. *The solution away from the critical levels*

Outside the critical layers, the unsteady flow is basically linear and inviscid. It is governed, to the order of approximation required in this study, e.g. terms involving $R^{-1}\bar{V}$ can be neglected, by the inviscid equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 , \tag{2.1}$$

$$2R^{-1}\frac{\partial u}{\partial \tau} + \bar{U}\frac{\partial u}{\partial x} + v\frac{\partial \bar{U}}{\partial y} = -\frac{\partial p}{\partial x} , \tag{2.2}$$

$$2R^{-1}\frac{\partial v}{\partial \tau} + \bar{U}\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y} , \tag{2.3}$$

$$2R^{-1}\frac{\partial w}{\partial \tau} + \bar{U}\frac{\partial w}{\partial x} = -\frac{\partial p}{\partial z} . \tag{2.4}$$

The elimination of pressure yields:

$$\left(2R^{-1}\frac{\partial}{\partial \tau} + \bar{U}\frac{\partial}{\partial x}\right)\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) - \frac{\partial \bar{U}}{\partial y}\frac{\partial w}{\partial x} = 0 , \tag{2.5}$$

and

$$\left(2R^{-1}\frac{\partial}{\partial \tau} + \bar{U}\frac{\partial}{\partial x}\right)\nabla^2 v - \bar{U}_{yy}\frac{\partial v}{\partial x} = 0 . \tag{2.6}$$

On introducing the multiple timescales referred to above, the time derivative needs to be transformed according to

$$\frac{\partial}{\partial \tau} \to R\frac{\partial}{\partial t} + \tfrac{1}{2}R\epsilon^{\frac{1}{3}}\frac{\partial}{\partial t_1} + \epsilon^{-\frac{1}{3}}\frac{\partial}{\partial \tau_1} + \frac{\partial}{\partial \tau} .$$

The velocity (u, v, w) and the pressure p of the disturbance are expanded as follows:

$$u = \epsilon u_1 + \epsilon^{\frac{4}{3}}u_2 + \dots , \tag{2.7}$$

$$v = \epsilon v_1 + \epsilon^{\frac{4}{3}}v_2 + \dots , \tag{2.8}$$

$$w = \epsilon w_1 + \epsilon^{\frac{4}{3}}w_2 + \dots , \tag{2.9}$$

$$p = \epsilon p_1 + \epsilon^{\frac{4}{3}}p_2 + \epsilon^{\frac{5}{3}}p_3 \dots . \tag{2.10}$$

The 'early time' linear solution is just a normal mode, so we seek solutions of the form

$$v_1 = A(t_1)\bar{v}_1(y)\cos\beta z E + \text{c.c.} , \tag{2.11}$$

where

$$E = \exp(\mathrm{i}\alpha x - \mathrm{i}\hat{\theta}(t)) , \tag{2.12}$$

$A(t_1)$ and $\hat{\theta}(t)$ are the amplitude and phase respectively of the disturbance, and α and β are the imposed streamwise and spanwise wavenumbers. In the case of a steady non-parallel shear layer the form of the solution has to be changed slightly so that the frequency and spanwise wavenumber are imposed (e.g. see GC and Appendix A).

As is standard in a WKBJLG analysis we write

$$\frac{d\hat{\theta}}{dt} = \tfrac{1}{2}\alpha c(\tau_0) + \tfrac{1}{2}\lambda^{\frac{1}{2}}\epsilon^{\frac{1}{2}}\Omega_1(\tau_0) + \dots , \tag{2.13}$$

where c is the local wavespeed, and Ω_1 represents a higher order correction to the local frequency which arises from a viscous sublayer adjacent to the wall (Cowley 1987). For simplicity we have assumed that the two oblique waves are of *equal* amplitude. In principle it is straightforward to extend the analysis to unequal amplitudes; however the asymmetry in the amplitudes complicates the algebra, especially in the viscous case. Note that as in Wu & Cowley (1993), the dependence on the slow timescales τ and τ_1 is parametric and will not be written out explicitly. The function $\bar{v}_1$ satisfies Rayleigh's equation

$$(\bar{U} - c)(D^2 - \bar{\alpha}^2)\bar{v}_1 - \bar{U}_{yy}\bar{v}_1 = 0 , \tag{2.14}$$

where

$$\bar{\alpha} = (\alpha^2 + \beta^2)^{\frac{1}{2}} .$$

The boundary conditions are that $\bar{v}_1 = 0$ on $y = 0$, and $\bar{v}_1 \to 0$ as $y \to \infty$.

We let $\eta = y - y_c^j$, where y_c^j is the jth critical level at which $\bar{U} = c$. Then as $\eta \to \pm 0$, $\bar{v}_1$ has the following asymptotic behaviour:

$$\bar{v}_1 \sim a_j^{\pm}\phi_a + b_j^{\pm}[\phi_b + p_j\phi_a \log|\eta|] , \tag{2.15}$$

where

$$\phi_a = \eta + \tfrac{1}{2}p_j\eta^2 + \dots , \qquad \text{and} \qquad \phi_b = 1 + q_j\eta^2 + \dots .$$

The function v_2 takes the form:

$$v_2 = \bar{v}_2(y, t_1)E\cos\beta z + \bar{v}_2^{(0,2)}\cos 2\beta z + \text{c.c.} + \dots , \tag{2.16}$$

where a relatively large longitudinal-vortex component, $\bar{v}_2^{(0,2)}\cos 2\beta z$, has had to be included in order to match to the inner solution. Further details concerning the origin of this term are given in §3, although a brief explanation can be given as follows. Near a critical level u_1 and w_1 are proportional to η^{-1}; see (2.28) and (2.29). Hence within the critical layers the leading-order velocity perturbations are of size $(u, v, w) \sim (\epsilon^{\frac{2}{3}}, \epsilon, \epsilon^{\frac{2}{3}})$. As a result of quadratic interactions within the z-momentum equation, an x-independent spanwise flow of $O(\epsilon)$ is forced in the critical layers (e.g. this comes from balancing the $\epsilon^{\frac{1}{3}}w_{t_1}$ and uw_x terms). From the continuity equation this forces an $O(\epsilon^{\frac{4}{3}})$ x-independent velocity normal to the wall. A detailed analysis of this term in §3 shows that it tends to a constant at the 'edge' of the critical layers; hence the x-independent term must be included in (2.16). Indeed there is a jump in the value of this term across the critical layers.

The function $\bar{v}_2$ is the deviation of the eigenfunction from its neutral state, and satisfies the inhomogeneous Rayleigh equation

$$\left[D^2 - \left(\bar{\alpha}^2 + \frac{\bar{U}_{yy}}{\bar{U} - c}\right)\right]\bar{v}_2 = (i\alpha)^{-1}\left\{\left[-\frac{dA}{dt_1} - (i\alpha\bar{U}_\tau\tau_1)A\right]\frac{\bar{U}_{yy}}{(\bar{U} - c)^2} + \frac{i\alpha\bar{U}_{yy\tau}\tau_1 A}{\bar{U} - c}\right\}\bar{v}_1 , \tag{2.17}$$

with the usual boundary conditions of $\bar{v}_2 = 0$ on $y = 0$ and $\bar{v}_2 \to 0$ as $y \to +\infty$. The asymptotic behaviour of $\bar{v}_2$ as $y \to y_c^j$ is

$$\begin{aligned} \bar{v}_2 \sim & -b_j^{\pm}r_j\log|\eta| + (a_j^{\pm}r_j + b_j^{\pm}s_j)\eta\log|\eta| + \dots \\ & + c_j^{\pm}\phi_a + d_j^{\pm}[\phi_b + p_j\phi_a\log|\eta|] , \end{aligned} \tag{2.18}$$

where

$$p_j = \frac{\bar{U}_{yy}}{\bar{U}_y} , \tag{2.19}$$

$$q_j = \tfrac{1}{2}\bar{\alpha}^2 + \frac{1}{2}\frac{\bar{U}_{yyy}}{\bar{U}_y} - \frac{\bar{U}_{yy}^2}{\bar{U}_y^2} , \tag{2.20}$$

$$r_j = (\mathrm{i}\alpha)^{-1}\frac{\bar{U}_{yy}}{\bar{U}_y^2}\left[-\frac{\mathrm{d}A}{\mathrm{d}t_1} - (\mathrm{i}\alpha\bar{U}_\tau\tau_1)A\right] , \tag{2.21}$$

$$s_j = (\mathrm{i}\alpha)^{-1}\left\{(-\mathrm{i}\alpha\bar{U}_{y\tau}\tau_1 A)\frac{\bar{U}_{yy}}{\bar{U}_y^2} + (\mathrm{i}\alpha\tau_1 A)\frac{\bar{U}_{yy\tau}}{\bar{U}_y}\right.$$
$$\left. + \left(-\frac{\mathrm{d}A}{\mathrm{d}t_1} - \mathrm{i}\alpha\bar{U}_\tau\tau_1 A\right)\frac{\bar{U}_y\bar{U}_{yyy} - \bar{U}_{yy}^2}{\bar{U}_y^3}\right\} . \tag{2.22}$$

Recall that all the basic-flow quantities are evaluated at time τ_0 and at the critical level y_c^j. The jumps $(a_j^+ - a_j^-)$, etc., will be determined by analysing the critical layers in § 3.

The continuity equation suggests that we write

$$w_1 = A\bar{w}_1 E \sin\beta z + \text{c.c.} ,$$

where $\bar{w}_1$ satisfies the equation

$$\frac{\partial\bar{w}_1}{\partial y} + \frac{\bar{U}_y}{\bar{U} - c}\bar{w}_1 = -\beta\bar{v}_1 .$$

This has the solution

$$\bar{w}_1 = \bar{\alpha}^{-1}\sin\theta\left(\frac{\bar{U}_y}{\bar{U} - c}\bar{v}_1 - \bar{v}_{1,y}\right) , \tag{2.23}$$

where

$$\sin\theta = \beta/\bar{\alpha} .$$

The velocity u_1 has the form

$$u_1 = A\bar{u}_1 E\cos\beta z + \bar{u}_1^{(0,2)}(y, t_1)\cos 2\beta z + \text{c.c.} , \tag{2.24}$$

where $\bar{u}_1$ is obtained from the continuity equation as

$$\bar{u}_1 = -(\mathrm{i}\alpha)^{-1}\left\{\left[\frac{\bar{U}_y}{\bar{U} - c}\bar{v}_1 - \bar{v}_{1,y}\right]\sin^2\theta + \bar{v}_{1,y}\right\} . \tag{2.25}$$

In order for u_1 to be able to match with the inner solution (see § 3), a spanwise-dependent mean flow, $\bar{u}_1^{(0,2)}(y, t_1)\cos 2\beta z$, has to be included at leading order (see also GC). As will be shown later, this mean flow is driven by a streamwise slip velocity across the critical layer, which itself is generated by a nonlinear interaction inside the critical layer. Although the mean flow is large in the sense that it has the same magnitude as the fundamental waves, it has no back effect on the critical-layer dynamics. Its dependence on t_1 can be viewed as forcing the longitudinal vortex represented by $\bar{v}_2^{(0,2)}$ and $\bar{w}_2^{(0,2)}$.

Similarly, we write the leading-order pressure perturbation as

$$p_1 = A(t_1)\bar{p}_1 E\cos\beta z + \text{c.c.} ,$$

where

$$\bar{p}_1 = \mathrm{i}\bar{\alpha}^{-1}\cos\theta[\bar{U}_y\bar{v}_1 - (\bar{U}-c)\bar{v}_{1,y}] \ . \tag{2.26}$$

As $y \to y_c^j$, the asymptotic solutions of $\bar{p}_1, \bar{u}_1, \bar{w}_1$ become

$$\bar{p}_1 \sim \mathrm{i}\bar{\alpha}^{-1}\bar{U}_y\cos\theta b_j^{\pm} + \dots \ , \tag{2.27}$$

$$\bar{u}_1 \sim -(\mathrm{i}\alpha)^{-1}\sin^2\theta b_j^{\pm}\eta^{-1} + \dots \ , \tag{2.28}$$

$$\bar{w}_1 \sim \bar{\alpha}^{-1}\sin\theta b_j^{\pm}\eta^{-1} + \dots \ . \tag{2.29}$$

Note that the singularity in $\bar{u}_1$ is a simple pole rather than the logarithmic branch point characteristic of a two-dimensional (singular) disturbance. As GC observed, it is this difference that results in the faster nonlinear evolutionary timescale compared with the corresponding two-dimensional case.

We now introduce an inner variable:

$$Y = \frac{\eta}{\epsilon^{\frac{1}{3}}} \ . \tag{2.30}$$

The outer expansions written in terms of this inner variable are then:

$$\begin{aligned} v \sim\ & \epsilon b_j^{\pm}AE\cos\beta z + \epsilon^{\frac{4}{3}}\log\epsilon^{\frac{1}{3}}(-b_j^{\pm}r_j + b_j^{\pm}p_jAY)E\cos\beta z \\ & + \epsilon^{\frac{4}{3}}[(-b_j^{\pm}r_j\log|Y| + d_j^{\pm}) + A(a_j^{\pm}Y + b_j^{\pm}p_jY\log|Y|)]E\cos\beta z \\ & + \epsilon^{\frac{5}{3}}\log\epsilon^{\frac{1}{3}}[(a_j^{\pm}r_j + b_j^{\pm}s_j + d_j^{\pm}p_j)Y + \frac{1}{2}Ap_jb_j^{\pm}Y^2]E\cos\beta z \\ & + \epsilon^{\frac{5}{3}}[c_j^{\pm}Y + (a_j^{\pm}r_j + b_j^{\pm}s_j + d_j^{\pm}p_j)Y\log|Y|E\cos\beta z + \text{c.c.} + \dots \ , \end{aligned} \tag{2.31}$$

$$u \sim \epsilon^{\frac{2}{3}}(-\mathrm{i}\alpha)^{-1}\sin^2\theta Ab_j^{\pm}Y^{-1}E\cos\beta z + \text{c.c.} + \dots \ , \tag{2.32}$$

$$w \sim \epsilon^{\frac{2}{3}}\bar{\alpha}^{-1}\sin\theta b_j^{\pm}AY^{-1}E\cos\beta z + \text{c.c.} + \dots \ , \tag{2.33}$$

$$p \sim \epsilon\ \mathrm{i}\bar{\alpha}^{-1}\bar{U}_y\cos\theta Ab_j^{\pm}E\cos\beta z + \text{c.c.} + \dots \ . \tag{2.34}$$

2.2. *Solvability condition*

By multiplying both sides of (2.17) by $\bar{v}_1$, and integrating from 0 to $+\infty$ with respect to y, we obtain

$$-\sum_j\left\{(\bar{v}_{2y}\bar{v}_1 - \bar{v}_{1y}\bar{v}_2)|_{y\to y_c^{j+}} - (\bar{v}_{2y}\bar{v}_1 - \bar{v}_{1y}\bar{v}_2)|_{y\to y_c^{j-}}\right\} = \mathrm{i}\alpha^{-1}J_1\frac{\mathrm{d}A}{\mathrm{d}t_1} + J_2\tau_1 A \ , \tag{2.35}$$

where J_1 and J_2 are formally defined by the following integrals respectively:

$$J_1 = \int_0^{+\infty}\frac{\bar{U}_{yy}}{(\bar{U}-c)^2}\bar{v}_1^2\,\mathrm{d}y \ , \tag{2.36}$$

$$J_2 = \int_0^{+\infty}\left[-\frac{\bar{U}_{yy}\bar{U}_\tau}{(\bar{U}-c)^2} + \frac{\bar{U}_{yy\tau}}{(\bar{U}-c)}\right]\bar{v}_1^2\,\mathrm{d}y \ . \tag{2.37}$$

Attention must be paid to the fact that both these integrals and the left-hand side of (2.35) are singular. On making use of the asymptotic solutions (2.15) and (2.18), and on analysing the singular behaviour of the integrals in the interval $[y_c^j - \hat{\epsilon}, y_c^j + \hat{\epsilon}]$ where $\hat{\epsilon}$ is a small number, we find that the singular parts on the left-hand side of (2.35) cancel those on the right-hand side, leaving the finite part to give the solvability condition for (2.17), namely

$$\mathrm{i}\alpha^{-1}J_1\frac{\mathrm{d}A}{\mathrm{d}t_1}+J_2\tau_1 A=-\sum_j\left\{(b_j^+c_j^+-b_j^-c_j^-)-r_j(b_j^+a_j^+-b_j^-a_j^-)\right.$$
$$\left.-p_j(b_j^+d_j^+-b_j^-d_j^-)-(a_j^+d_j^+-a_j^-d_j^-)\right\}\ , \qquad (2.38)$$

where the sum is over all critical layers. An examination of the singular terms in (2.35) shows that the cancellations occur in such a manner that J_1 and J_2 can be interpreted as Hadamard finite parts (Hickernell 1984).

After the jumps $(a_j^+-a_j^-)$, etc., are determined in the next section, the amplitude equation can be derived from (2.38). The nonlinearity is introduced into the amplitude equation through the jumps; thus for the purpose of deriving the amplitude equation, we only need to consider those parts of the inner solutions contributing to the jumps. This consideration simplifies the algebra to a certain extent.

3 Inner expansion

Equations (2.31)–(2.34) suggest that the inner expansions within the jth critical layer take the following form:

$$u=\epsilon^{\frac{2}{3}}U_1+\epsilon^{\frac{3}{3}}U_2+\epsilon^{\frac{4}{3}}U_3+\dots\ , \qquad (3.1)$$

$$v=\epsilon^{\frac{3}{3}}V_1+\epsilon^{\frac{4}{3}}V_2+\epsilon^{\frac{5}{3}}V_3+\dots\ , \qquad (3.2)$$

$$w=\epsilon^{\frac{2}{3}}W_1+\epsilon^{\frac{3}{3}}W_2+\epsilon^{\frac{4}{3}}W_3+\dots\ , \qquad (3.3)$$

$$p=\epsilon^{\frac{3}{3}}P_1+\epsilon^{\frac{4}{3}}P_2+\epsilon^{\frac{5}{3}}P_3+\dots\ , \qquad (3.4)$$

where $O(\epsilon^n\log\epsilon^{\frac{1}{3}})$ terms have not been explicitly included. This is because as far as deriving the amplitude equation is concerned, they are passive in the sense that they match onto the outer solutions automatically whenever the solutions at $O(\epsilon^n)$ match, i.e. matching at $O(\epsilon^n\log\epsilon^{\frac{1}{3}})$ does not yield any additional jump conditions. Of course, these terms must be included if a quantitative comparison between theory and experiment is to be made.

The function V_1 satisfies the equation

$$\mathrm{L}_0\frac{\partial^2 V_1}{\partial Y^2}=0\ , \qquad (3.5)$$

where

$$\mathrm{L}_0=\frac{\partial}{\partial t_1}+(\bar{U}_y Y+\bar{U}_\tau\tau_1)\frac{\partial}{\partial x}-\lambda\frac{\partial^2}{\partial Y^2}\ . \qquad (3.6)$$

The solution which matches the outer expansion is

$$V_1=\hat{A}(t_1)E\cos\beta z+\text{c.c.}\ , \qquad (3.7)$$

where $\hat{A}=b_jA$, and $b_j^+=b_j^-=b_j$, i.e. the jump $(b_j^+-b_j^-)$ is zero.

The expansion of the y-momentum equation gives

$$\frac{\partial P_1}{\partial Y}=0\ , \qquad (3.8)$$

and so the appropriate solution is

$$P_1=\mathrm{i}\alpha^{-1}\bar{U}_y\cos\theta\hat{A}E\cos\beta z+\text{c.c.} \qquad (3.9)$$

From the z-momentum equation we have that

$$\mathrm{L}_0 W_1 = -\frac{\partial P_1}{\partial z} . \tag{3.10}$$

We let $W_1 = \hat{W}_1 E \sin\beta z + \text{c.c.}$, then $\hat{W}$ satisfies

$$\hat{\mathrm{L}}_0^{(1)} \hat{W}_1 = \mathrm{i}\bar{U}_y \sin\theta\cos\theta\hat{A} , \tag{3.11}$$

where

$$\hat{\mathrm{L}}_0^{(n)} = \frac{\partial}{\partial t_1} + n\mathrm{i}\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1) - \lambda\frac{\partial^2}{\partial Y^2} . \tag{3.12}$$

Equation (3.11) can be solved using Fourier transforms to yield the solution

$$\hat{W}_1 = \mathrm{i}\bar{U}_y \sin\theta\cos\theta\hat{W}_0^{(0)} , \tag{3.13}$$

where

$$\hat{W}_0^{(n)} = \int_0^{+\infty} \xi^n \hat{A}(t_1-\xi)\mathrm{e}^{-s_1\xi^3-\mathrm{i}\Omega\xi}\mathrm{d}\xi , \tag{3.14}$$

and

$$\Omega = \alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1) , \qquad s_1 = \tfrac{1}{3}\lambda\alpha^2\bar{U}_y^2 . \tag{3.15}$$

Similarly, the leading-order streamwise velocity $\hat{U}_1$ can be written as

$$U_1 = \hat{U}_1 E \cos\beta z + \text{c.c.}$$

It follows from the continuity equation that

$$\hat{U}_1 = -\bar{U}_y \sin^2\theta\hat{W}_0^{(0)} . \tag{3.16}$$

At $O(\epsilon^{\frac{4}{3}})$, V_2 satisfies

$$\mathrm{L}_0 V_{2,YY} = \mathrm{L}_1 V_1 + \frac{\partial}{\partial Y}\left[\frac{\partial S_{11}}{\partial x} + \frac{\partial S_{31}}{\partial z}\right] , \tag{3.17}$$

where

$$\mathrm{L}_1 = -\left\{(\tfrac{1}{2}\bar{U}_{yy}Y^2 + \bar{U}_{y\tau}\tau_1 Y + \tfrac{1}{2}\bar{U}_{\tau\tau}\tau_1^2)\frac{\partial}{\partial x} + \lambda\frac{\partial}{\partial \tau_1}\right\}\frac{\partial^2}{\partial Y^2} + \bar{U}_{yy}\frac{\partial}{\partial x} , \tag{3.18}$$

and S_{11}, S_{31} are Reynolds stresses defined by

$$S_{11} = \frac{\partial U_1^2}{\partial x} + \frac{\partial U_1 V_1}{\partial Y} + \frac{\partial U_1 W_1}{\partial z} , \tag{3.19}$$

$$S_{31} = \frac{\partial U_1 W_1}{\partial x} + \frac{\partial V_1 W_1}{\partial Y} + \frac{\partial W_1^2}{\partial z} . \tag{3.20}$$

S_{11} and S_{31} can be rewritten as

$$S_{11} = S_{11}^{(0,0)} + S_{11}^{(0,2)}\cos 2\beta z + S_{11}^{(2,0)}E^2 + S_{11}^{(2,2)}E^2\cos 2\beta z + \text{c.c.} , \tag{3.21}$$

$$S_{31} = S_{31}^{(0,2)}\sin 2\beta z + S_{31}^{(2,2)}E^2\sin 2\beta z + \text{c.c.} \tag{3.22}$$

After some calculation, we find that

$$S_{11}^{(0,0)} = \tfrac{1}{2}\mathrm{i}\alpha\bar{U}_y^2\sin^2\theta\hat{A}^*\hat{W}_0^{(1)} , \tag{3.23}$$

$$S_{11}^{(0,2)} = \tfrac{1}{2}\mathrm{i}\alpha\bar{U}_y^2\sin^2\theta\hat{A}^*\hat{W}_0^{(1)} , \tag{3.24}$$

$$S_{11}^{(2,0)} = \tfrac{1}{2}\mathrm{i}\alpha\bar{U}_y^2\sin^2\theta[\hat{A}\hat{W}_0^{(1)} + 2\sin^2\theta\hat{W}_0^{(0)}\hat{W}_0^{(0)}] , \tag{3.25}$$

$$S_{11}^{(2,2)} = \tfrac{1}{2}\mathrm{i}\alpha\bar{U}_y^2\sin^2\theta\hat{A}\hat{W}_0^{(1)} , \tag{3.26}$$

$$S_{31}^{(0,2)} = \tfrac{1}{2}\beta\bar{U}_y^2\cos^2\theta[\hat{A}^*\hat{W}_0^{(1)} + 2\sin^2\theta\hat{W}_0^{(0)}\hat{W}_0^{*(0)}] , \tag{3.27}$$

$$S_{31}^{(2,2)} = \tfrac{1}{2}\beta\bar{U}_y^2\cos^2\theta\hat{A}\hat{W}_0^{(1)} . \tag{3.28}$$

From inspection of (3.21), (3.22) and the right-hand side of (3.17), we conclude that V_2 has a solution of the form

$$V_2 = \hat{V}_2^{(1)}E\cos\beta z + \hat{V}_2^{(0,2)}\cos 2\beta z + \hat{V}_2^{(2,0)}E^2 + \text{c.c.} \tag{3.29}$$

The fundamental component $\hat{V}_2^{(1)}$ is driven only by the linear forcing term, i.e. $\mathrm{L}_1 V_1 = \mathrm{i}\alpha\bar{U}_{yy}\hat{A}$. This is exactly the same as in the two-dimensional case (e.g. see Wu & Cowley 1993). By analogy, we obtain the jump conditions

$$a_j^+ - a_j^- = \pi\mathrm{i}p_j b_j \mathrm{sgn}(\bar{U}_y) , \tag{3.30}$$

$$d_j^+ - d_j^- = -\pi\mathrm{i}r_j b_j \mathrm{sgn}(\bar{U}_y) . \tag{3.31}$$

Substituting (3.23)–(3.28) into (3.17), we find that the functions $\hat{V}_2^{(0,2)}$ and $\hat{V}_2^{(2,0)}$ satisfy

$$\left\{\frac{\partial}{\partial t_1} - \lambda\frac{\partial^2}{\partial Y^2}\right\}\hat{V}_{2,YY}^{(0,2)} = -\mathrm{i}\bar{S}^3\sin^2\theta[\hat{A}^*\hat{W}_0^{(2)} + 4\sin^2\theta\hat{W}_0^{*(0)}\hat{W}_0^{(1)}] , \tag{3.32}$$

$$\hat{\mathrm{L}}_\lambda^{(2)}\hat{V}_{2,YY}^{(2,0)} = \mathrm{i}\bar{S}^3\sin^2\theta[\hat{A}\hat{W}_0^{(2)} + 4\sin^2\theta\hat{W}_0^{(0)}\hat{W}_0^{(1)}] , \tag{3.33}$$

respectively, where we have put

$$\bar{S} = \alpha\bar{U}_y.$$

Inserting (3.14) into (3.32) and (3.33), and integrating these equations, we find the solutions

$$\hat{V}_{2,YY}^{(0,2)} = -\mathrm{i}\bar{S}^3\sin^2\theta\int_0^{+\infty}\int_0^{+\infty}\xi I_v^{(0,2)}(\xi,\eta)\hat{A}^*(t_1-\eta)\hat{A}(t_1-\eta-\xi)\mathrm{e}^{-\mathrm{i}\Omega\xi}\mathrm{d}\xi\mathrm{d}\eta , \tag{3.34}$$

$$\hat{V}_{2,YY}^{(2,0)} = \mathrm{i}\bar{S}^3\sin^2\theta\int_0^{+\infty}\int_0^{+\infty}I_v^{(2,0)}(\xi,\eta)\hat{A}(t_1-\eta)\hat{A}(t_1-\eta-\xi)\mathrm{e}^{-\mathrm{i}\Omega(\xi+2\eta)}\mathrm{d}\xi\mathrm{d}\eta , \tag{3.35}$$

where we have defined

$$I_v^{(0,2)}(\xi,\eta) = I^{(0)}(\xi,\eta)[\xi + 4\sin^2\theta\int_0^\eta \mathrm{e}^{-2s_1\zeta^3-3s_1\xi\zeta^2}\mathrm{d}\zeta] , \tag{3.36}$$

$$I_v^{(2,0)}(\xi,\eta) = I^{(1)}(\xi,\eta)[\xi^2 + 4\sin^2\theta\int_0^\eta(\xi+2\zeta)\mathrm{e}^{2s_1\zeta^3+3s_1\xi\zeta^2}\mathrm{d}\zeta] , \tag{3.37}$$

$$I^{(0)}(\xi,\eta) = \mathrm{e}^{-s_1(\xi^3+3\xi^2\eta)} , \tag{3.38}$$

$$I^{(1)}(\xi,\eta) = \mathrm{e}^{-s_1(\xi^3+3\xi^2\eta+6\xi\eta^2+4\eta^3)} . \tag{3.39}$$

In simplifying $\hat{V}_{2,YY}^{(0,2)}$ into the present compact form, we have dropped a purely imaginary part; from (3.29) this does not alter the physical velocity. A similar procedure will be followed on solving for $\hat{W}^{(0,2)}$ later on.

It follows from (3.34) that as $Y \to \pm\infty$

$$(\hat{V}_2^{(0,2)} + \text{c.c.}) \sim \pm 8\bar{S}\sin^4\theta\pi\int_0^{+\infty}\int_0^\eta \mathrm{e}^{-2s_1\zeta^3}|\hat{A}(t_1-\eta)|^2\mathrm{d}\zeta\mathrm{d}\eta$$
$$+\{\text{order-one 'no-jump' terms}\} + o(1) .$$

We conclude that a longitudinal vortex component must be included in the $O(\epsilon^{\frac{4}{3}})$ outer expansion of v so that a match with $\hat{V}_2^{(0,2)}$ can be achieved.

The function U_2 satisfies

$$\mathrm{L}_0 U_{2,Y} = -\bar{U}_{yy}V_1 + F_1(Y)\frac{\partial W_1}{\partial z} - F_2(Y)\frac{\partial^2 U_1}{\partial x \partial Y} - \frac{\partial}{\partial Y}S_{11} + \bar{U}_y\frac{\partial W_2}{\partial z} , \tag{3.40}$$

where

$$F_1(Y) = \bar{U}_{yy}Y + \bar{U}_{Y\tau}\tau_1 , \tag{3.41}$$

and

$$F_2(Y) = \tfrac{1}{2}\bar{U}_{yy}Y^2 + \bar{U}_{y\tau}\tau_1 Y + \tfrac{1}{2}\bar{U}_{\tau\tau}\tau_1^2 . \tag{3.42}$$

The solution has the form

$$U_2 = \hat{U}_2^{(1)}E\cos\beta z + \hat{U}_2^{(0,0)} + \hat{U}_2^{(0,2)}\cos 2\beta z + \hat{U}_2^{(2,0)}E^2 + \hat{U}_2^{(2,2)}E^2\cos 2\beta z + \text{c.c.} \tag{3.43}$$

In order to derive the amplitude equation, we need only the mean-flow distortions $\hat{U}_2^{(0,0)}$ and $\hat{U}_2^{(0,2)}$. These satisfy the following equations respectively:

$$\left[\frac{\partial}{\partial t_1} - \lambda\frac{\partial^2}{\partial Y^2}\right]\hat{U}_{2,Y}^{(0,0)} = -\tfrac{1}{2}\alpha^{-1}\bar{S}^3\sin^2\theta \hat{A}^*\hat{W}_0^{(2)} , \tag{3.44}$$

$$\left[\frac{\partial}{\partial t_1} - \lambda\frac{\partial^2}{\partial Y^2}\right]\hat{U}_{2,Y}^{(0,2)} = -\frac{\partial}{\partial Y}S_{11}^{(0,2)} - \bar{U}_y\hat{V}_{2,Y}^{(0,2)} , \tag{3.45}$$

where $S_{11}^{(0,2)}$ is defined by (3.24). The solutions are

$$\hat{U}_{2,Y}^{(0,0)} = -\tfrac{1}{2}\alpha^{-1}\bar{S}^3\sin^2\theta\int_0^{+\infty}\int_0^{+\infty}\xi^2 I^{(0)}(\xi,\eta)\hat{A}^*(t_1-\eta)\hat{A}(t_1-\eta-\xi)\mathrm{e}^{-\mathrm{i}\Omega\xi}\mathrm{d}\xi\mathrm{d}\eta \tag{3.46}$$

$$\hat{U}_{2,Y}^{(0,2)} = -\tfrac{1}{2}\alpha^{-1}\bar{S}^3\sin^2\theta\int_0^{+\infty}\int_0^{+\infty} I_u^{(0,2)}(\xi,\eta)\hat{A}^*(t_1-\eta)\hat{A}(t_1-\eta-\xi)e^{-\mathrm{i}\Omega\xi}\mathrm{d}\xi\mathrm{d}\eta \tag{3.47}$$

where we have put

$$I_u^{(0,2)}(\xi,\eta) = I^{(0)}(\xi,\eta)\left[\xi^2+2\xi\eta+4\sin^2\theta\int_0^{\eta}2(\eta-\zeta)\mathrm{e}^{-2s_1\zeta^3-3s_1\xi\zeta^2}\mathrm{d}\zeta\right] . \tag{3.48}$$

Integrating $\hat{U}_{2,Y}^{(0,2)}$ once with respect to Y, we find that as $Y \to \pm\infty$

$$\begin{aligned}(\hat{U}_2^{(0,2)} + \text{c.c.}) \sim \mp 8\alpha^{-1}\bar{S}^2\sin^4\theta\pi\int_0^{+\infty}\int_0^{\eta}(\eta-\zeta)\mathrm{e}^{-2s_1\zeta^3}|\hat{A}(t_1-\eta)|^2\mathrm{d}\zeta\mathrm{d}\eta \\ +\{\text{order-one `no-jump' terms}\} + o(1) .\end{aligned} \tag{3.49}$$

This is in fact a streamwise slip velocity generated by nonlinear interactions inside the critical layers. The outer expansion for u must match to this; hence the leading-order outer expansion for u, (2.24), must contain a component representing a spanwise-dependent mean flow.

The spanwise velocity W_2 satisfies the following equation:

$$\mathrm{L}_0 W_2 = -\frac{\partial P_2}{\partial z} - F_2(Y)\frac{\partial W_1}{\partial x} - S_{31} , \tag{3.50}$$

where F_2 is defined by (3.42). The solution W_2 has the form

$$W_2 = \hat{W}_2^{(1)}E\sin\beta z + \hat{W}_2^{(0,2)}\sin 2\beta z + \hat{W}_2^{(2,2)}E^2\sin 2\beta z + \text{c.c.} \tag{3.51}$$

Since at next order we are only interested in the interaction that generates the fundamental, $\hat{W}_2^{(1)}$ does not need to be calculated. Moreover, we find that as far as deriving the amplitude equation is concerned, it is sufficient to solve just for the mean component $\hat{W}_2^{(0,2)}$. It follows from the continuity equation that

$$\hat{W}_2^{(0,2)} = -(2\beta)^{-1}\hat{V}_{2,Y}^{(0,2)} , \tag{3.52}$$

and hence from (3.52) and (3.35) that as $Y \to \pm\infty$

$$(\hat{W}^{(0,2)} + \text{c.c.}) \sim 4\sin^4\theta\beta^{-1}Y^{-2}\int_0^{+\infty}\int_0^{\eta} \mathrm{e}^{-2s_1\zeta^3}|\hat{A}(t_1-\eta)|^2 \mathrm{d}\zeta\,\mathrm{d}\eta + o(Y^{-2}) . \tag{3.53}$$

An important point to note is that all the spanwise velocity components generated by the nonlinear interactions are bounded as $Y \to \pm\infty$. This is in contrast to the case of a purely viscous critical layer where unbounded growth of these components can occur (e.g. see Hall & Smith 1991). This difference arises because the inclusion of unsteadiness in our critical layers means that as $Y \to \pm\infty$ the balance is between the unsteady inertial term and the nonlinear forcing terms. In a steady viscous critical layer the balance is between the viscous diffusion term and the nonlinear forcing terms. For instance, in the case of $\hat{W}^{(0,2)}$ the balance is between $\hat{W}_{YY}^{(0,2)}$ (in our notation) and a nonlinear forcing which decays like Y^{-2} as $Y \to \pm\infty$. Therefore $\hat{W}^{(0,2)}$ grows like '$\pm Y + \log Y$'. In a wave/vortex interaction, this unbounded growth is one of the reasons why small-amplitude three-dimensional disturbances are able to generate order-one mean-flow distortions (Hall & Smith 1991).

The unbounded growth of a mean-flow distortion away from a viscous critical layer has been noted in other stability problems, e.g. for two-dimensional disturbances in stratified shear flows (Churilov & Shukhman 1987*b*; see also Goldstein & Hultgren 1988). However, often the evolution of the disturbance is sufficiently rapid that a linear diffusion layer is established between the critical layer and the outer region to eliminate this growth (see also Brown & Stewartson 1978; Haynes & Cowley 1986, and §4.4). In the nonlinear Rayleigh-wave/vortex interaction model of Hall & Smith (1991), the diffusion layer merges with the outer region because the amplitude evolution is sufficiently slow – in our notation Hall & Smith (1991) effectively have *two* timescales, namely the time t to describe the rapid oscillation of the disturbance, and the time τ to describe both the evolution of the basic flow and the growth of the disturbance. The relationship of the present work to the wave/vortex interaction approach will be examined in §4.4.

We now proceed to derive the amplitude equation. For this purpose, it is sufficient to seek the solution for V_3 only. This term satisfies

$$\mathrm{L}_0 V_{3,YY} = \mathrm{L}_1 V_2 + \mathrm{L}_2 V_1 + \frac{\partial}{\partial Y}\left[\frac{\partial}{\partial x}S_{12} + \frac{\partial}{\partial z}S_{32}\right] + \dots , \tag{3.54}$$

where

$$\begin{aligned}\mathrm{L}_2 = &-\left[\tfrac{1}{6}\bar{U}_{yyy}Y^3 + \bar{U}_{yy\tau}\tau_1 Y^2 + \bar{U}_{y\tau\tau}\tau_1^2 Y + \tfrac{1}{6}\bar{U}_{\tau\tau\tau}\tau_1^3\right]\frac{\partial^3}{\partial x \partial Y^2} \\ &+ \left[\bar{U}_{yyy}Y + \bar{U}_{yy\tau}\tau_1\right]\frac{\partial}{\partial x} \\ &- \left[\frac{\partial}{\partial t_1} + (\bar{U}_y Y + \bar{U}_\tau \tau_1)\frac{\partial}{\partial x}\right]\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right] ,\end{aligned} \tag{3.55}$$

the Reynolds stresses S_{12} and S_{32} are defined by

$$S_{12} = \frac{\partial}{\partial x}(2U_1U_2) + \frac{\partial}{\partial Y}(U_1V_2 + U_2V_1) + \frac{\partial}{\partial z}(U_1W_2 + U_2W_1) , \tag{3.56}$$

$$S_{32} = \frac{\partial}{\partial x}(U_1W_2 + U_2W_1) + \frac{\partial}{\partial Y}(W_1V_2 + W_2V_1) + \frac{\partial}{\partial z}(2W_1W_2) , \tag{3.57}$$

and for brevity only those terms that contribute to the jumps across the critical layers are explicitly included.

At this order it is only necessary to find the fundamental component, i.e.

$$V_3 = \hat{V}_3 E \cos\beta z + \text{c.c.} + \dots .$$

Similarly we write

$$\frac{\partial}{\partial x}S_{12} + \frac{\partial}{\partial z}S_{32} = \bar{M}E\cos\beta z + \text{c.c.} + \dots ,$$

with the non-fundamental components being omitted.

We note that the relevant part of the linear forcing term, i.e. $(L_1V_2 + L_2V_1)$, is the same as $F^{(l)}(Y,t_1)E$ in Wu & Cowley (1993). Thus the solution forced by it, denoted here by $\hat{V}_3^{(l)}E$, has the same asymptotic behaviour:

$$\hat{V}^{(l)}_{3,Y} \sim (a_j^+p_j + 2q_jb_j + \tfrac{1}{2}p_j^2b_j)Y + (a_j^+r_j + p_jd_j^+ + s_jb_j)\log|Y| \\ +\{\pm\tfrac{1}{2}\pi \mathrm{isng}(\bar{U}_y)(a_j^+r_j + p_jd_j^+ + s_jb_j) + \dots\} . \tag{3.58}$$

Also, in order to aid the calculation of the solutions and their asymptotic behaviour, it proves convenient to write the nonlinear forcing term $\bar{M}$ as a sum of the four terms, namely

$$\bar{M} = \bar{M}_1 + \bar{M}_2 + \bar{M}_3 + \bar{M}_0 ,$$

where

$$\bar{M}_1 = \mathrm{i}\alpha\hat{A}\hat{U}^{*(0,0)}_{2,Y} + \tfrac{1}{2}i\alpha\hat{A}\hat{U}^{*(0,2)}_{2,Y} + \tfrac{1}{2}\beta\hat{A}\hat{W}^{*(0,2)}_{2,Y} , \tag{3.59}$$

$$\bar{M}_2 = \mathrm{i}\alpha\hat{U}_{1,Y}\hat{V}_2^{*(0,2)} - 2\alpha^2\hat{U}_1\hat{U}_2^{*(0,2)} , \tag{3.60}$$

$$\bar{M}_3 = 2\mathrm{i}\alpha\hat{U}^*_{1,Y}\hat{V}_2^{(2,0)} , \tag{3.61}$$

$$\bar{M}_0 = \mathrm{i}\alpha\hat{A}\hat{U}^{(0,0)}_{2,Y} + \tfrac{1}{2}\mathrm{i}\alpha\hat{A}\hat{U}^{(0,2)}_{2,Y} + \tfrac{1}{2}\beta\hat{A}\hat{W}^{(0,2)}_{2,Y} + \mathrm{i}\alpha\hat{A}^*\hat{U}_2^{(2,0)} \\ +\tfrac{1}{2}\alpha\hat{A}^*\hat{U}_2^{(2,2)} + \tfrac{1}{2}\beta\hat{A}^*\hat{W}_2^{(2,2)} + \mathrm{i}\alpha\hat{U}_{1,Y}\hat{V}_2^{(0,2)} - 2\alpha^2\hat{U}_1\hat{U}_2^{(0,2)} . \tag{3.62}$$

We now let $\hat{V}_3^{(j)}$ denote the solution driven by $\bar{M}_j$ $(j = 0,1,2,3)$. Then

$$\hat{V}_{3,YY} = \hat{V}^{(l)}_{3,YY} + \hat{V}^{(1)}_{3,YY} + \hat{V}^{(2)}_{3,YY} + \hat{V}^{(3)}_{3,YY} + \hat{V}^{(0)}_{3,YY} , \tag{3.63}$$

where

$$\hat{L}_0^{(1)}\hat{V}^{(j)}_{3,YY} = \bar{M}_{j,Y} , \tag{3.64}$$

and $\hat{L}_0^{(1)}$ is defined by (3.12). In addition we denote the Fourier transform of $\hat{V}^{(j)}_{3,YY}$ by $\Gamma_j(k)$:

$$\Gamma_j(k) = \int_{-\infty}^{+\infty} \hat{V}_{3,YY}\mathrm{e}^{-ikY}\mathrm{d}Y .$$

To obtain the jump we need to evaluate $\{\hat{V}^{(j)}_{3,Y}(+\infty) - \hat{V}^{(j)}_{3,Y}(-\infty)\}$, or equivalently

$$\Gamma_j(0) = \hat{V}^{(j)}_{3,Y}(+\infty) - \hat{V}^{(j)}_{3,Y}(-\infty) . \tag{3.65}$$

A calculation shows that $\hat{V}_3^{(0)}$ makes no contribution to the jump.

The forcing $\bar{M}_1$ represents the Reynolds stress generated by the interaction between the vertical velocity of the fundamental wave and the induced spanwise flow. Using (3.65), and solving (3.64) with $j=1$, we find that

$$\hat{V}^{(1)}_{3,Y}(+\infty)-\hat{V}^{(1)}_{3,Y}(-\infty)$$
$$=\bar{S}^4 j_0\sin^2\theta\int_0^{+\infty}\int_0^{+\infty}\tilde{K}^{(1)}_v(\xi,\eta)\hat{A}(t_1-\xi)\hat{A}(t_1-\eta-\xi)\hat{A}^*(t_1-2\xi-\eta)\mathrm{d}\xi\mathrm{d}\eta\ , \tag{3.66}$$

where we have put

$$j_0=\pi|\bar{S}|^{-1}\ ,$$

$$\tilde{K}^{(1)}_v(\xi,\eta)=\tilde{K}^{(0)}(\xi,\eta)\left\{2\xi^3+\xi^2\eta+2\sin^2\theta\int_0^{\eta}[\xi^2+2\xi(\eta-\zeta)]\mathrm{e}^{-2s_1\zeta^3-3s_1\xi\zeta^2}\mathrm{d}\zeta\right\}\ , \tag{3.67}$$

and

$$\tilde{K}^{(0)}(\xi,\eta)=\mathrm{e}^{-s_1(2\xi^3+3\xi^2\eta)}\ . \tag{3.68}$$

The forcing $\bar{M}_2$ represents the interaction between the streamwise velocity of the fundamental wave and the induced spanwise flow. In order to overcome a technical difficulty in evaluating the asymptotic form of $\hat{V}^{(2)}_{3,Y}$ we write

$$\hat{V}^{(2)}_{3,YY}=Q_2(Y,t_1)-\mathrm{i}\alpha\bar{U}_y^{-1}\hat{U}_{1,YY}\hat{U}_2^{*(0,2)}\ . \tag{3.69}$$

Substituting this and (3.60) into (3.64), we find that $Q_2(Y,t_1)$ satisfies

$$\hat{\mathrm{L}}^{(1)}_0 Q_2=\tilde{N}_2(Y,t_1)\ , \tag{3.70}$$

where

$$\tilde{N}_2=\bar{M}_{2,Y}+\mathrm{i}\alpha\bar{U}_y^{-1}\hat{\mathrm{L}}^{(1)}_0(\hat{U}_{1,YY}\hat{U}_2^{(0,2)})\ .$$

Since

$$\hat{\mathrm{L}}^{(1)}_0\hat{U}_{1,Y}=-\mathrm{i}\alpha\bar{U}_y\hat{U}_1\ , \tag{3.71}$$

we obtain, by differentiating with respect to Y, that

$$\hat{\mathrm{L}}^{(1)}_0\hat{U}_{1,YY}=-2\mathrm{i}\alpha\bar{U}_y\hat{U}_{1,Y}\ . \tag{3.72}$$

From use of (3.72) and the complex conjugate of (3.45), we find that

$$\tilde{N}_2=\mathrm{i}\alpha\hat{U}_{1,Y}\hat{V}^{*(0,2)}_{2,Y}-2\alpha^2\hat{U}_1\hat{U}^{*(0,2)}_{2,Y}-2\mathrm{i}\lambda\alpha\bar{U}_y^{-1}\hat{U}_{1,YYY}\hat{U}^{*(0,2)}_{2,Y}-\mathrm{i}\alpha\bar{U}_y^{-1}\hat{U}_{1,YY}S^{*(0,2)}_{11}\ . \tag{3.73}$$

Then after solving for $Q_2(Y,t_1)$ from (3.70) using Fourier transforms, we obtain

$$\hat{V}^{(2)}_{3,Y}(+\infty)-\hat{V}^{(2)}_{3,Y}(-\infty)$$
$$=2\bar{S}^4 j_0\sin^4\theta\int_0^{+\infty}\int_0^{+\infty}\tilde{K}^{(2)}_v(\xi,\eta)\hat{A}(t_1-\xi)\hat{A}(t_1-\eta-\xi)\hat{A}^*(t_1-2\xi-\eta)\mathrm{d}\xi\mathrm{d}\eta\ , \tag{3.74}$$

where we have put

$$\begin{aligned}\tilde{K}^{(2)}_v(\xi,\eta)=&\left\{\tilde{K}^{(0)}(\xi,\eta)\int_0^{\xi}[\zeta(2\eta+3\zeta)-\xi(\xi+2\eta+2\zeta)]\mathrm{e}^{-3s_1\xi\zeta^2}\mathrm{d}\zeta\right.\\&\left.+\tilde{K}^{(1)}(\xi,\eta)\int_0^{\xi}[(\eta+\zeta)(\eta+3\zeta)-(\xi+\eta)(\xi+\eta+2\zeta)]\mathrm{e}^{-3s_1(\xi+\eta)(2\eta+\zeta)\zeta}\mathrm{d}\zeta\right\}\\&+4\sin^2\theta\left\{\tilde{K}^{(0)}(\xi,\eta)\int_0^{\xi}\mathrm{d}\zeta\mathrm{e}^{-3s_1\xi\zeta^2}\int_0^{\eta+\zeta}(\upsilon-\eta-\zeta)[1+2\lambda\bar{S}^2(\xi-\zeta)\zeta^2]\mathrm{e}^{-s_1(2\upsilon^3+3\xi\upsilon^2)}\mathrm{d}\upsilon\right.\\&\left.+\tilde{K}^{(1)}(\xi,\eta)\int_0^{\xi}\mathrm{d}\zeta\mathrm{e}^{-3s_1(\xi+\eta)(2\eta+\zeta)\zeta}\int_0^{\zeta}(\upsilon-\zeta)[1+2\lambda\bar{S}^2(\xi-\zeta)(\eta+\zeta)^2]\mathrm{e}^{-s_1[2\upsilon^3+3(\xi+\eta)\upsilon^2]}\mathrm{d}\upsilon\right\},\end{aligned} \tag{3.75}$$

and

$$\tilde{K}^{(1)}(\xi,\eta) = e^{-s_1[\xi^3+\eta^3+(\xi+\eta)^3]} . \quad (3.76)$$

The forcing $\bar{M}_3$ is the Reynolds stress generated by the interaction between the streamwise velocity of the three-dimensional fundamental and the 'two-dimensional harmonic' generated by the interaction of the fundamentals. On writing

$$\hat{V}^{(3)}_{3,YY} = Q_3(Y,t_1)+\bar{U}_y^{-1}\hat{U}^*_{1,YY}\hat{V}^{(2,0)}_{2,Y} , \quad (3.77)$$

and substituting this and (3.60) into (3.64), it can be shown that Q_3 satisfies

$$\hat{\mathrm{L}}_0^{(1)}Q_3 = \tilde{N}_3(Y,t_1) , \quad (3.78)$$

where

$$\tilde{N}_3 = \bar{M}_{3,Y} - \bar{U}_y^{-1}\hat{L}_0^{(1)}(\hat{U}^*_{1,YY}\hat{V}^{(2,0)}_{2,Y}) .$$

Observing that

$$\hat{\mathrm{L}}_0^{(2)}\hat{V}^{(2,0)}_{2,Y} = 2\mathrm{i}\alpha\bar{U}_y\hat{V}^{(2,0)}_2+2\mathrm{i}\alpha S^{(2,0)}_{11} ,$$

and using the complex conjugate of equation (3.72), we find that

$$\tilde{N}_3 = -2\mathrm{i}\alpha\bar{U}_y^{-1}\hat{U}^*_{1,YY}S^{*(2,0)}_{11}+2\lambda\bar{U}_y^{-1}\hat{U}^*_{1,YYY}\hat{V}^{(2,0)}_{2,YY} , \quad (3.79)$$

where $S^{(2,0)}_{11}$ is defined by (3.25). After solving for Q_3 from (3.78), it can be shown that

$$\hat{V}^{(3)}_{3,Y}(+\infty)-\hat{V}^{(3)}_{3,Y}(-\infty)$$
$$= 4\bar{S}^4 j_0\sin^4\theta\int_0^{+\infty}\int_0^{+\infty}\tilde{K}^{(3)}_v(\xi,\eta)\hat{A}(t_1-\xi)\hat{A}(t_1-\eta-\xi)\hat{A}^*(t_1-2\xi-\eta)\mathrm{d}\xi\mathrm{d}\eta , \quad (3.80)$$

where

$$\tilde{K}^{(3)}_v(\xi,\eta) = \tilde{K}^{(1)}(\xi,\eta)\left\{\int_0^{\xi}\eta\zeta[1+2\lambda\bar{S}^2(\xi+\eta+\zeta)^2(\xi-\zeta)]\,\Pi_0(\xi,\eta,\zeta)\mathrm{d}\zeta\right.$$
$$\left.+\,4\sin^2\theta\int_0^{\xi}\mathrm{d}\zeta\,\Pi_0(\xi,\eta,\zeta)\int_0^{\zeta}(\zeta-v)[1+2\lambda\bar{S}^2(\xi+\eta+\zeta)^2(\xi-\zeta)]e^{s_1(2v^3+3\eta v^2)}\mathrm{d}v\right\}, \quad (3.81)$$

and

$$\Pi_0(\xi,\eta,\zeta) = e^{-s_1(4\zeta^3+6\xi\zeta^2+9\eta\zeta^2+6\xi\eta\zeta+6\eta^2\zeta)} . \quad (3.82)$$

By matching $\hat{V}_{3,Y}$ with the outer expansion we find that

$$c_j^+-c_j^- = \hat{V}_{3,Y}(+\infty)-\hat{V}_{3,Y}(-\infty) . \quad (3.83)$$

Combining (3.64), (3.58), (3.66), (3.74), (3.80), together with (3.67), (3.75) and (3.81), we conclude that

$$c_j^+-c_j^- = \pi\mathrm{i}\,\mathrm{sgn}(\bar{U}_y)(a_j^+r_j+p_jd_j^++s_jb_j)$$
$$+\,\pi\alpha^3|\bar{U}_y|^3\sin^2\theta\int_0^{+\infty}\int_0^{+\infty}K_j(\xi,\eta|\lambda)\hat{A}(t_1-\xi)\hat{A}(t_1-\xi-\eta)\hat{A}^*(t_1-2\xi-\eta)\mathrm{d}\xi\mathrm{d}\eta, \quad (3.84)$$

where

$$
\begin{aligned}
K_j(\xi,\eta|\lambda) &= \tilde{K}^{(0)}(\xi,\eta)(2\xi^3+\xi^2\eta) \\
&+2\sin^2\theta\left\{\tilde{K}^{(0)}(\xi,\eta)\int_0^\eta [\xi^2+2\xi(\eta-\zeta)]\mathrm{e}^{-2s_1\zeta^3-3s_1\xi\zeta^2}\,\mathrm{d}\zeta\right. \\
&+\tilde{K}^{(0)}(\xi,\eta)\int_0^\xi [\zeta(2\eta+3\zeta)-\xi(\xi+2\eta+2\zeta)]\mathrm{e}^{-3s_1\xi\zeta^2}\,\mathrm{d}\zeta \\
&+2\tilde{K}^{(1)}(\xi,\eta)\int_0^\xi \eta\zeta[1+6s_1(\xi-\zeta)(\xi+\eta+\zeta)^2]\Pi_0(\xi,\eta,\zeta)\,\mathrm{d}\zeta \\
&\left.+\tilde{K}^{(1)}(\xi,\eta)\int_0^\xi [(\eta+\zeta)(\eta+3\zeta)-(\xi+\eta)(\xi+\eta+2\zeta)]\mathrm{e}^{-3s_1(\xi+\eta)(2\eta+\zeta)\zeta}\,\mathrm{d}\zeta\right\} \\
&+8\sin^4\theta\left\{\tilde{K}^{(0)}(\xi,\eta)\int_0^\xi \mathrm{d}\zeta\,\mathrm{e}^{-3s_1\xi\zeta^2}\int_0^{\eta+\zeta}(v-\eta-\zeta)[1+6s_1(\xi-\zeta)\zeta^2]\mathrm{e}^{-s_1(2v^3+3\xi v^2)}\,\mathrm{d}v\right. \\
&+2\tilde{K}^{(1)}(\xi,\eta)\int_0^\xi \mathrm{d}\zeta\,\Pi_0(\xi,\eta,\zeta)\int_0^\zeta(\zeta-v)[1+6s_1(\xi-\zeta)(\xi+\eta+\zeta)^2]\mathrm{e}^{s_1(2v^3+3\eta v^2)}\,\mathrm{d}v \\
&\left.+\tilde{K}^{(1)}(\xi,\eta)\int_0^\xi \mathrm{d}\zeta\,\mathrm{e}^{-3s_1(\xi+\eta)(2\eta+\zeta)\zeta}\int_0^\zeta(v-\zeta)[1+6s_1(\xi-\zeta)(\eta+\zeta)^2]\mathrm{e}^{-s_1[2v^3+3(\xi+\eta)v^2]}\,\mathrm{d}v\right\}.
\end{aligned}
\tag{3.85}
$$

Here the suffix j refers to the jth critical layer, $\tilde{K}^{(0)}$, $\tilde{K}^{(1)}$ and Π_0 are defined by (3.68), (3.76) and (3.82) respectively, and the dependence on λ is through s_1 (see (3.15)). Although the kernel $K_j(\xi,\eta|\lambda)$ is algebraically complicated, nevertheless it simplifies to the following form when $\lambda = 0$:

$$K(\xi,\eta) = (2\xi^3+\xi^2\eta)-2\sin^2\theta(2\xi^3-\xi\eta^2)-4\sin^4\theta(\xi^2\eta+\xi\eta^2)\,. \tag{3.86}$$

This is just the kernel obtained for the inviscid case (GC; Wu 1991).

We note that both $(a_j^+-a_j^-)$ and $(d_j^+-d_j^-)$ correspond to the classical $\pm\pi$ phase shift in the outer expansion, while $(c_j^+-c_j^-)$ is modified by nonlinearity. It is through this modification that nonlinear effects control the evolution of the disturbance.

3.1. *The amplitude evolution equation*

By inserting the jumps (3.30), (3.31) and (3.84) into (2.38), we obtain the amplitude equation

$$\frac{\mathrm{d}A}{\mathrm{d}t_1} = g_0\tau_1 A+\int_0^{+\infty}\int_0^{+\infty}\sum g_j K_j(\xi,\eta|\lambda)A(t_1-\xi)A(t_1-\xi-\eta)A^*(t_1-2\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,, \tag{3.87}$$

where the sum is over all critical layers; in the case of the Stokes layer there are two for most of the right-hand branch of curve A of figure 1. The kernel $K_j(\xi,\eta|\lambda)$ is defined by (3.85), while

$$g_0 = f_0/f\,, \qquad g_j = f_j/f\,, \tag{3.88}$$

and

$$f_j = -\pi\alpha^3\sin^2\theta b_j^2|b_j|^2|\bar{U}_y|^3\,. \tag{3.89}$$

The constants f and f_0 are the same as in the two-dimensional case (Wu & Cowley 1993), namely

$$f = \mathrm{i}\alpha^{-1}\left\{\sum_j \pi b_j\left[2\mathrm{i}\frac{\bar{U}_{yy}}{|\bar{U}_y|^2}a_j^+ + \mathrm{i}b_j\frac{\bar{U}_y\bar{U}_{yyy}-\bar{U}_{yy}^2}{\bar{U}_y|\bar{U}_y|^2} + b_j\pi\frac{\bar{U}_{yy}^2}{\bar{U}_y^3}\right] + J_1\right\}, \tag{3.90}$$

$$f_0 = \sum_j\left\{2\pi\mathrm{i}b_j\frac{\bar{U}_{yy}\bar{U}_\tau}{|\bar{U}_y|^2}a_j^+ - \pi\mathrm{i}b_j^2\left[\frac{\bar{U}_{yy\tau}}{|\bar{U}_y|} - \frac{\bar{U}_{yy}\bar{U}_{y\tau}}{|\bar{U}_y|^2} - \frac{(\bar{U}_y\bar{U}_{yyy}-\bar{U}_{yy}^2)\bar{U}_\tau}{\bar{U}_y|\bar{U}_y|^2}\right]\right.$$
$$\left. + \pi^2 b_j^2\frac{\bar{U}_{yy}\bar{U}_\tau}{\bar{U}_y^3}\right\} - J_2, \tag{3.91}$$

while J_1 and J_2 are defined by (2.36) and (2.37) respectively. In Appendix A we indicate the minor modifications that are necessary to the above amplitude equation in order to consider the viscous spatial evolution of disturbances in free shear layers (cf. GC).

In the case of the Stokes layer the coefficients g_j are evaluated. For instance, in figures 3(a) and 3(b) they have been plotted against the wavenumber $\bar{\alpha} = (\alpha^2+\beta^2)^{\frac{1}{2}}$ for the right-hand branch of curve A. The plot of the scaled coefficient ($g_0 \sec\theta$) against $\bar{\alpha}$ is the same as shown in figure 3 of Wu & Cowley (1993) (provided that α there is replaced by $\bar{\alpha}$). For our purposes here it is sufficient to recall that the real part of g_0 is always negative.

In the inviscid limit, $\lambda = 0$, the amplitude equation becomes

$$\frac{\mathrm{d}A}{\mathrm{d}t_1} = g_0\tau_1 A + g\int_0^{+\infty}\int_0^{+\infty} K(\xi,\eta)A(t_1-\xi)A(t_1-\xi-\eta)A^*(t_1-2\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,, \tag{3.92}$$

where the kernel $K(\xi,\eta)$ is defined by (3.86),

$$g = \sum_j g_j\,, \tag{3.93}$$

and the sum is over all critical layers. Note that although we have assumed that $\bar{U}_{yy}(y_c) \neq 0$, the nonlinear kernel $K(\xi,\eta)$ is exactly the same as that of GC. This is because nonlinear interactions inside the critical layers are only associated with the pole singularity in u and w, compared with which the logarithmic branch-point singularity associated with $\bar{U}_{yy}(y_c) \neq 0$ is much weaker.

In the spirit of Stewartson & Stuart (1972), Churilov & Shukhman (1988), GC and others, we require solutions to (3.87) to match with the exponentially growing linear stage as $t_1 \to -\infty$, i.e.

$$A \to A_0 e^{g_0\tau_1 t_1} \qquad \text{as} \qquad t_1 \to -\infty\,. \tag{3.94}$$

Following GC, the parameters A_0 and τ_1 can be scaled out by introducing the rescaled variables

$$\bar{A} = A e^{-\mathrm{i}(\bar{T}_0 + g_{0i}\tau_1 t_1)}|g|^{\frac{1}{2}}/(g_{0r}\tau_1)^3\,, \tag{3.95}$$

$$\bar{t} = g_{0r}\tau_1 t_1 - \bar{t}_0\,, \tag{3.96}$$

$$\bar{\lambda} = \lambda/(g_{0r}\tau_1)^3\,, \tag{3.97}$$

where g_{0r} and g_{0i} are the real and imaginary parts of g_0 respectively, and $\bar{T}_0$ and $\bar{t}_0$ are chosen so that

$$-\bar{t}_0 + \mathrm{i}\bar{T}_0 = \log[A_0|g|^{\frac{1}{2}}/(g_{0r}\tau_1)^3]\,. \tag{3.98}$$

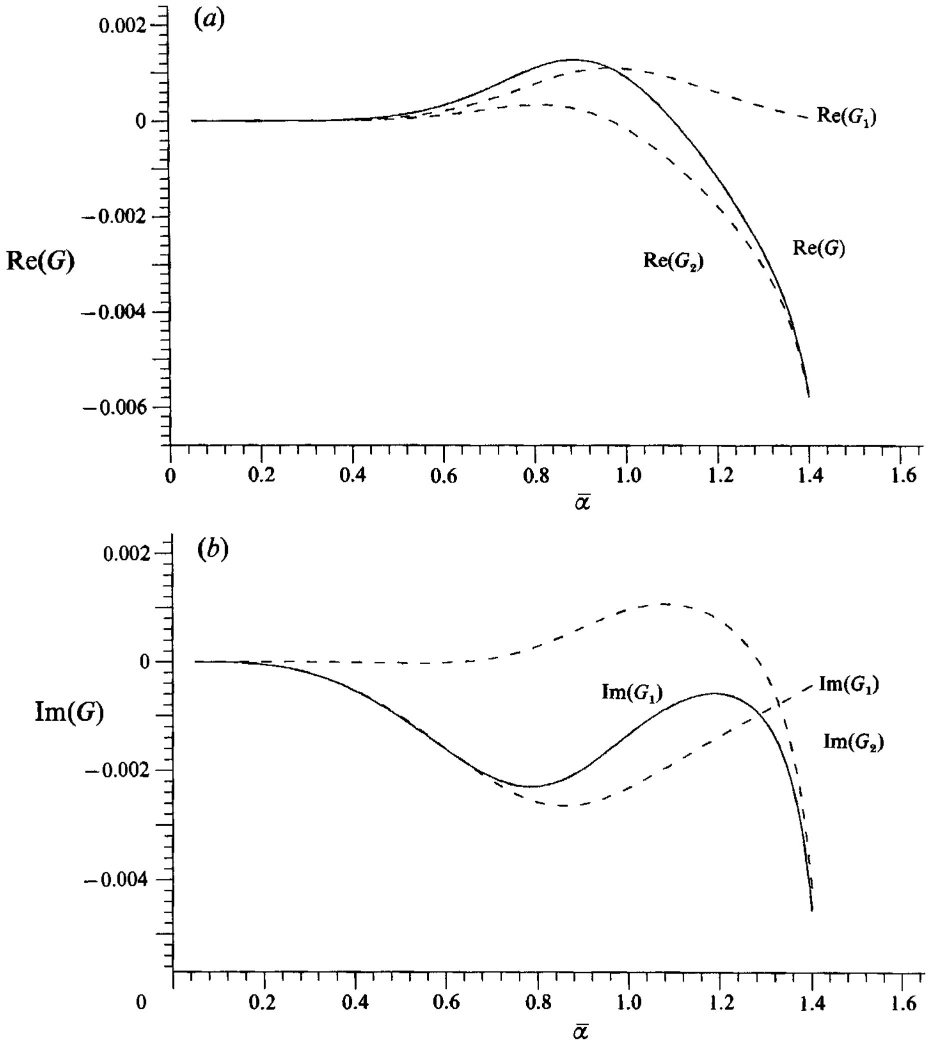

FIGURE 3. The scaled coefficients in the amplitude equation (3.87), where $G_j = g_j/[\alpha^4 \sin^2\theta](j = 1, 2)$, and $G = G_1 + G_2$. (*a*) The real parts. (*b*) The imaginary parts.

The evolution equation and the asymptotic behaviour then become

$$\frac{\mathrm{d}\bar{A}}{\mathrm{d}\bar{t}} = \bar{A} + \int_0^{+\infty}\int_0^{+\infty}\sum_j v_j K_j(\xi,\eta|\lambda)\bar{A}(\bar{t}-\xi)\bar{A}(\bar{t}-\xi-\zeta)\bar{A}^*(\bar{t}-2\xi-\zeta)\mathrm{d}\xi\mathrm{d}\zeta\ , \tag{3.99}$$

$$\bar{A} \rightarrow \mathrm{e}^{\bar{t}} \quad \text{as} \quad \bar{t} \rightarrow -\infty\ , \tag{3.100}$$

where $v_j = g_j/|g|$, and we have written $\bar{\lambda}$ in the kernel as λ.

If there is only one critical layer, then $|v_1| \equiv |v| = 1$. Thus in this case the amplitude equation depends only on λ, θ and $\arg v$. In this sense, the rescaling (3.95)–(3.97) achieves the same purpose as Shukhman's (1991) introduction of a 'logarithmic time'.

4 Study of the amplitude equation

4.1. *Finite-time singularity structure*

A singularity structure for the amplitude equation (3.92), i.e. the inviscid limit of (3.87), was proposed by GC in their study of the spatial evolution of disturbances in a free shear layer. They showed numerically that solutions developed a singularity at a finite distance downstream, or in terms of our temporal evolution problem, the solutions blew up within a *finite time*. The singularity proposed was

$$\bar{A} \sim \frac{a_0}{(t_s-\bar{t})^{3+i\sigma}} \qquad \text{as} \qquad \bar{t} \to t_s \, , \tag{4.1}$$

where σ and a_0 are real and complex constants respectively. Although this finite-time singularity was identified for the inviscid case, substitution of (4.1) into (3.99) shows that it is unaltered at leading order by viscous effects. We obtain

$$3+i\sigma = \sum_j v_j D_0 |a_0|^2 \, , \tag{4.2}$$

where

$$D_0 = \int_0^{+\infty}\int_0^{+\infty} K(\xi,\eta)[(1+\xi)(1+\xi+\eta)]^{-(3+i\sigma)}(1+2\xi+\eta)^{-(3-i\sigma)}\mathrm{d}\xi\mathrm{d}\eta \, , \tag{4.3}$$

and $|a_0|$ and σ can be solved from (4.2). The singularity time t_s can be determined numerically in the same way as described in Wu & Cowley (1993).

4.2. *Viscous limit*

Possibly the most surprising result of this paper is that the amplitude equation (3.87), or equivalently (3.99), does not admit an equilibrium solution. This is a significant difference from the two-dimensional viscous case (e.g. Goldstein & Leib 1989; Wu & Cowley 1993). The reason for this is that the integral of the kernel

$$\int_0^{+\infty}\int_0^{+\infty} K_j(\xi,\eta|\lambda)\mathrm{d}\xi\mathrm{d}\eta$$

does not exist.

In order to shed light on this observation, we now turn to examining the amplitude equation under the very viscous limit $\lambda \to +\infty$. This corresponds to the situation where nonlinear effects become important at times relatively close to a linear neutral curve; i.e. times such that $|\Delta\tau| \ll R^{-\frac{1}{3}}$ in terms of the notation of § 1. The growth rate of the instability waves is then relatively small, and the effects of viscosity are relatively large. In Appendix B it is shown that in this limit, the amplitude equation (3.87) can be reduced to

$$\lambda^{-\frac{1}{3}}\frac{\mathrm{d}A}{\mathrm{d}\bar{t}_1} = g_0\tau_1 A + \lambda^{-1}\bar{g}A\int_0^{+\infty}|A(\bar{t}_1-\eta)|^2\mathrm{d}\eta + O(\lambda^{-\frac{4}{3}}A^3) \, , \tag{4.4}$$

where $\bar{t}_1$ is defined in (B7). The complex constant $\bar{g}$ is

$$\bar{g} = \frac{2^{\frac{2}{3}}}{9}\sin^2\theta[1-2\sin^2\theta]\Gamma(\tfrac{1}{3})\sum_j g_j\beta_j^{-\frac{4}{3}} \, , \tag{4.5}$$

where the sum is again over all critical layers. The parameter λ can be scaled from the leading-order equation by the transformation

$$A = \lambda^{\frac{1}{3}}\breve{A} \, , \quad \text{and} \quad \tau_1 = \lambda^{-\frac{1}{3}}\breve{\tau}_1 \, , \tag{4.6}$$

to obtain

$$\frac{\mathrm{d}\breve{A}}{\mathrm{d}\bar{t}_1} = g_0\breve{\tau}_1\breve{A} + \bar{g}\breve{A}\int_0^{+\infty} |\breve{A}(\bar{t}_1-\eta)|^2\mathrm{d}\eta \ . \tag{4.7}$$

We note that in this limit the critical layers are of width $O(\lambda^{\frac{1}{3}}\epsilon^{\frac{1}{3}})$, i.e. $O(R^{-\frac{1}{3}})$, and are thus viscous at leading order. However, sandwiching each critical layer there are thicker diffusion layers of width $O(\lambda^{\frac{2}{3}}\epsilon^{\frac{1}{3}})$, i.e. $O(\lambda^{\frac{1}{3}}R^{-\frac{1}{3}})$, where unsteadiness effects are still important at leading order (cf. Brown & Stewartson 1978). From (2.8), (2.11) and (4.6) we observe that the mode amplitude at which nonlinear effects come into play, i.e. $\epsilon\lambda^{\frac{1}{3}} = \lambda^{-\frac{2}{3}}R^{-1}$, decreases as λ increases.

Since the completion of this study, Smith & Blennerhassett (1992) have published a derivation of essentially the same equation as (4.7) in the context of the lower-branch of the Tollmien–Schlichting instability of channel flow. Again the presence of diffusion layers is a feature of the analysis. However, rather than matching their solutions to an exponentially growing linear mode as $\bar{t}_1 \to -\infty$, they assume that a disturbance of suitable form is instantaneously introduced at $\bar{t}_1 = 0$. Recently Mankbadi, Wu & Lee (1993) and Wu (1993*b*) have shown that an extension of equation (4.7) arises in analyses of upper-branch boundary-layer stability. The integral term arises due to the presence of diffusion layers of the same type as here; indeed such a term appears to be a characteristic of flows where a diffusion layer is needed to accommodate the interaction between the three-dimensional waves.

As already noted, a remarkable feature of (4.7) is that the nonlinear term is non-local so that the amplitude equation is not of Stuart–Watson–Landau type.† In fact the nonlinear effect comes only from the second terms on the right-hand sides of (3.59) and (3.60) which represent the interaction between the fundamental and the induced mean-flow distortion — the higher harmonics play no role at leading order. This is in contrast to the case when $\lambda = O(1)$, where both the mean-flow distortion and the higher harmonics contribute. Note also that when $\theta = \frac{1}{4}\pi$, the coefficient $\bar{g}$ is zero so that the nonlinear term vanishes (see (4.5)). The same behaviour also occurs in the inviscid case (GC). However, when $0 < \lambda < \infty$, $K_j(\xi,\eta|\lambda)$ is non-zero even when $\theta = \frac{1}{4}\pi$.

Although a numerical investigation of (3.99) seems necessary, we note that (4.7) can be solved analytically. First we scale out various constants by a transformation similar to (3.95)–(3.96):

$$B = \breve{A}\mathrm{e}^{-\mathrm{i}(\breve{T}_{10}+g_{0i}\breve{\tau}_1\bar{t}_1)}|\bar{g}_r|^{\frac{1}{2}}/(g_{0r}\breve{\tau}_1) \ , \tag{4.8}$$

$$\breve{t}_1 = g_{0r}\breve{\tau}_1\bar{t}_1 - \breve{t}_{10} \ , \tag{4.9}$$

where $\breve{T}_{10}$ and $\breve{t}_{10}$ are chosen so that

$$-\breve{t}_{10} + \mathrm{i}\breve{T}_{10} = \log[\breve{A}_0|\bar{g}_r|^{\frac{1}{2}}/(g_{0r}\breve{\tau}_1)] \ , \tag{4.10}$$

and $\bar{g}_r$ is the real part of $\bar{g}$, i.e.

$$\bar{g}_r = \frac{2^{\frac{2}{3}}}{9}\sin^2\theta\cos 2\theta\,\Gamma(\tfrac{1}{3})\mathrm{Re}\{\sum_j g_j\beta_j^{-\frac{4}{3}}\} \ . \tag{4.11}$$

† The authors are grateful to Professor S.N. Brown for discussions without which the authors would not have realized this point.

The evolution equation (4.7), and the asymptotic behaviour, then become

$$\frac{\mathrm{d}B}{\mathrm{d}\check{t}_1} = B + \frac{\bar{g}}{|\bar{g}_r|}B\int_0^{+\infty} |B(\check{t}_1-\eta)|^2\mathrm{d}\eta\ , \tag{4.12}$$

and

$$B \to \exp(\check{t}_1) \qquad \text{as} \qquad \check{t}_1 \to -\infty\ . \tag{4.13}$$

By multiplying (4.12) by B^*, and then taking the complex conjugate, it is straightforward to show that

$$\frac{\mathrm{d}|B|^2}{\mathrm{d}\check{t}_1} = 2|B|^2 + \frac{2\bar{g}_r}{|\bar{g}_r|}|B|^2\int_0^{+\infty} |B(\check{t}_1-\eta)|^2\mathrm{d}\eta\ . \tag{4.14}$$

On solving a differentiated version of (4.14) subject to the initial condition (4.13) we obtain

$$|B|^2 = \frac{16\exp(2\check{t}_1)}{(4-(\bar{g}_r/|\bar{g}_r|)\exp(2\check{t}_1))^2}\ . \tag{4.15}$$

From (4.15) it is clear that the sign of $\bar{g}_r$ plays an important role in determining the terminal behaviour of solutions to (4.7). In particular, the solution develops a finite-time singularity when $\bar{g}_r > 0$, but decays exponentially at large times when $\bar{g}_r < 0$; when $\bar{g}_r = 0$ the solution for $|A|$ grows exponentially. Since (4.7) is a limiting form of (3.87), this *suggests* that when $\bar{g}_r > 0$, solutions to (3.87) or (3.99) will develop a finite-time singularity no matter how large λ is. However, when $\bar{g}_r < 0$ it seems likely that solutions will terminate in a finite-time singularity if λ is not too large, but will decay exponentially once λ exceeds a critical value. Our numerical results demonstrate that this is indeed the case.

4.3. *Numerical study of the amplitude equation*

We have integrated the rescaled amplitude equation (3.99) using a finite-difference method. Two independent schemes have been used as a check: a Milne's (predictor-corrector) method and an Adams–Moulton (implicit) method. Both schemes have sixth-order accuracy. The kernel $K_j(\xi,\eta|\lambda)$ is evaluated numerically using Simpson's rule.

We assume that $\bar{A}$ can be approximated by the linear solution $e^{\bar{t}}$ when $\bar{t} \leq -T_0$, where T_0 is a 'big' positive number; a suitable choice is determined by trial and error. The integral over the infinite domain $D = [0,+\infty) \times [0,+\infty)$ is approximated by that over a large but finite domain $D_0 = [0, X_0] \times [0, Y_0]$. For the viscous case, the tail over the domain $D_1 = (D - D_0)$ is neglected. This is justifiable because the factor

$$K_j(\xi,\eta|\lambda)\bar{A}(\bar{t}_1-\xi)\bar{A}(\bar{t}_1-\xi-\eta)\bar{A}(\bar{t}_1-2\xi-\eta)$$

decays exponentially as ξ and/or η tend to $+\infty$. Different values of X_0 and Y_0 were tried in the program before deciding on suitably large values; we find that it is sufficient to take $X_0 = Y_0 = \bar{t} + T_0$. For the inviscid case, we approximate the tail over D_1 analytically using the linear solution. However, we find that dropping this tail has little effect on our results.

The inviscid version of (3.99) has been studied by GC. Their numerical results confirm the singularity structure (4.1). Here we integrate the inviscid amplitude equation using the coefficients calculated for the Stokes layer. Figure 4 shows results for the wavenumber $\bar{\alpha} = 1.2$ on the right-hand branch of curve A. Four propagation angles θ were investigated: $\theta = 15°$, $30°$, $60°$ and $75°$. For $\theta = 15°$ and $30°$ the amplitudes exhibit an oscillatory behaviour, indicating a periodic energy exchange

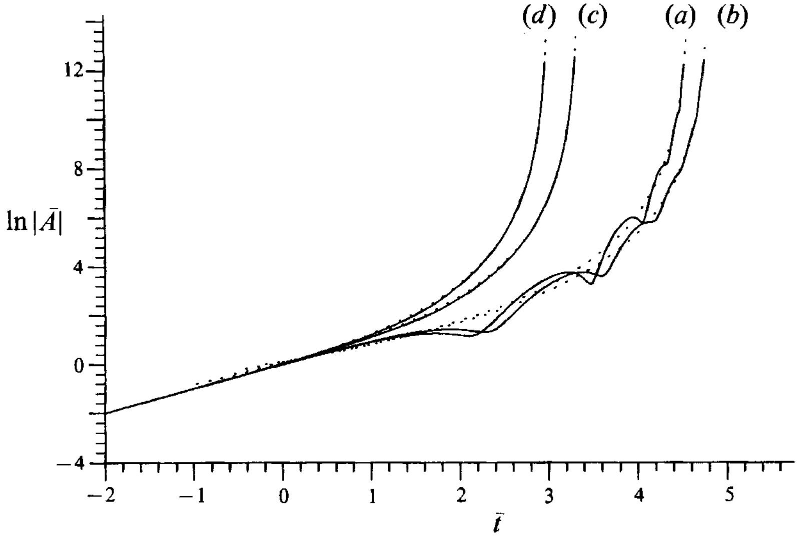

FIGURE 4. Ln$|\bar{A}|$ *vs.* the scaled time $\bar{t}$ for $\bar{\alpha} = 1.2$ and $\lambda = 0$ (inviscid limit): (*a*) $\theta = 15°$; (*b*) $\theta = 30°$; (*c*) $\theta = 60°$, (*d*) $\theta = 75°$. Solid lines: numerical solutions; dotted lines: local asymptotic solutions (4.1).

between the disturbance and the basic flow. Similar behaviour has been observed elsewhere (e.g. see GC and references therein). Local singular solutions are displayed as dotted lines. They show that a finite-time singularity occurs. Moreover, for $\theta = 60°$ and 75°, the local singular solutions fit rather well with the corresponding numerical results over a substantial range of time, even though they were expected to be valid only near the singularity time. For the inviscid case, we have also worked out a power-series solution of the form $\sum a_n e^{(2n+1)\bar{t}}$. The recursion relation for the a_n is given in Appendix C. Although this form of solution is strictly only valid when $\bar{t} \to -\infty$, it is found to have a rather sizeable range of validity when truncated at high order, say 40–50th order. This provides a check on the numerical results (see also Wu 1991). For $\theta = 60°$ and 75° the power-series solutions were able to reproduce the first oscillatory cycle.

We note that at $\theta = 60°$, a resonant triad interaction can occur if the initial disturbance includes an $O(\epsilon^{\frac{4}{3}})$ two-dimensional eigenmode with a wavenumber 2α and a phase velocity c (Goldstein & Lee, 1992; Wu, 1992). However, in the present problem this resonance does not occur. This is because such a two-dimensional mode is not present in our initial disturbance, and when excited by nonlinear quadratic interactions, it only has a magnitude of $O(\epsilon^2)$ — weaker than the $O(\epsilon^{\frac{4}{3}})$ strength required for a resonant interaction.

We now move onto the viscous case. As illustrated in figure 5 for $\theta = 60°$ and a wavenumber $\bar{\alpha} = 1.2$ on the right-hand branch of curve A, we find that increasing the viscosity generally delays the occurrence of the singularity. Note that $\bar{g}_r$ is positive for these parameter values, so the singularity cannot be eliminated no matter how large λ is. Figure 6 shows results for $\theta = 30°$, and the same wavenumber $\bar{\alpha} = 1.2$; the parameter $\bar{g}_r$ is now negative. As can be seen, viscosity delays the time to singularity formation if λ is not too large. However, once λ exceeds a critical value† (between

† No attempt has been made to determine the critical value precisely because it is very CPU intensive to integrate the amplitude equation for λ close to its critical value.

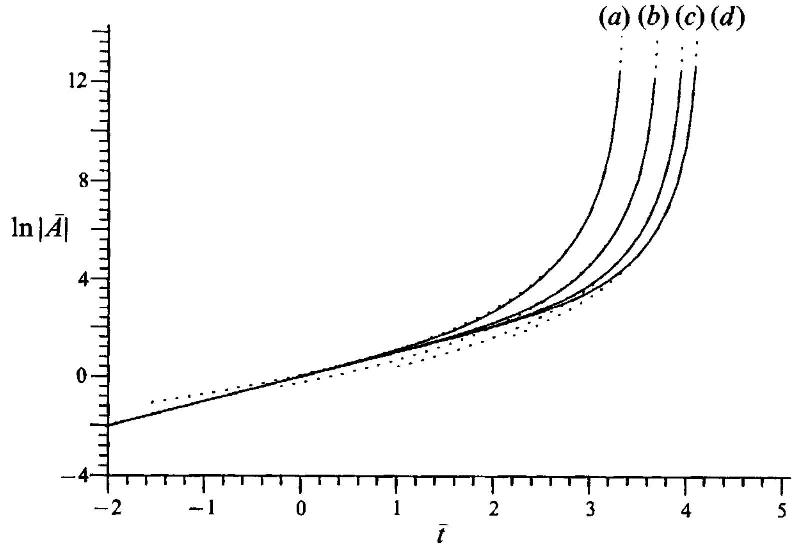

FIGURE 5. Ln$|\bar{A}|$ *vs.* the scaled time $\bar{t}$ for $\bar{\alpha} = 1.2$ and $\theta = 60°$: (*a*) $\lambda = 0$; (*b*) $\lambda = 10$; (*c*) $\lambda = 30$; (*d*) $\lambda = 50$. Solid lines: numerical solutions; dotted lines: local asymptotic solutions (4.1).

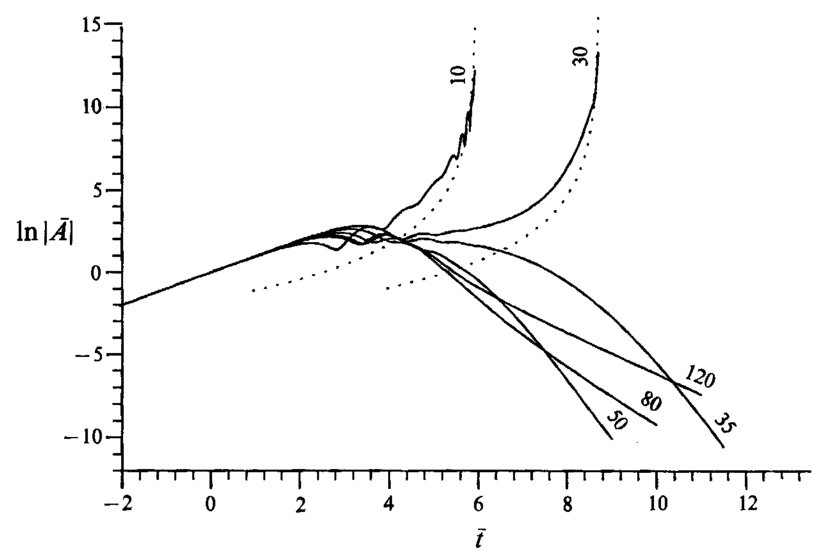

FIGURE 6. Ln$|\bar{A}|$ *vs.* the scaled time $\bar{t}$ for $\bar{\alpha} = 1.2$ and $\theta = 30°$. $\lambda = 10$, 30, 35, 50, 80, and 120. Solid lines: numerical solutions; dotted lines: local asymptotic solutions (4.1).

30.0 and 35.0 in this case), the amplitude decays exponentially, i.e. at large times $\log|\bar{A}| \sim -c_0\bar{t}$ or $|\bar{A}| \sim \mathrm{e}^{-c_0\bar{t}}$, where $c_0 > 0$ is a constant. This is significantly different from the two-dimensional case, where viscosity causes the disturbance to saturate at a finite amplitude.

Surprisingly, although the waves decay exponentially, in Appendix D we show that the 'slip' velocity, (D 1), generated by accumulated nonlinear effects *grows* linearly with time. This is illustrated in figure 7, where we plot a suitably scaled streamwise velocity jump. It follows that in the outer region the *spanwise-dependent* mean flow driven by

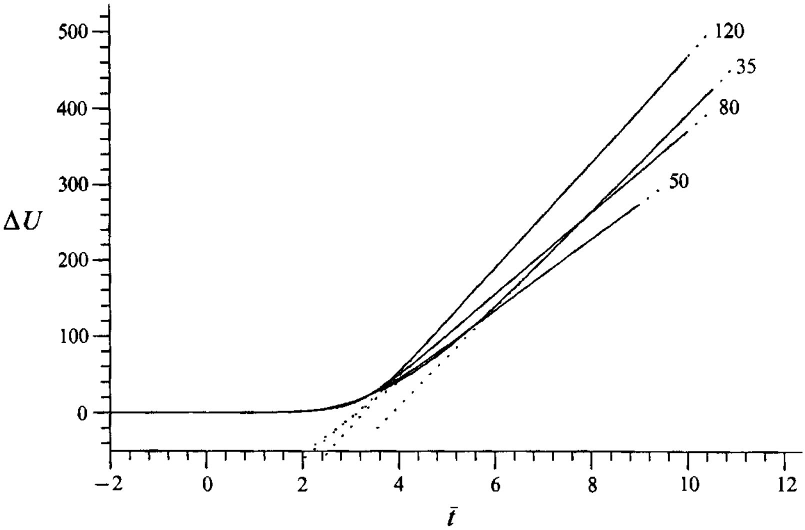

FIGURE 7. 'Vortex sheet' development: ΔU *vs.* the scaled time $\bar{t}$. Parameters: $\bar{\alpha} = 1.2$, $\theta = 30°$, and $\lambda = 35$, 50, 80 and 120. Solid lines: numerical solutions; dotted lines: asymptotic slopes.

the slip velocity grows linearly with time, while the longitudinal vortex equilibrates at a finite amplitude (see also Smith & Blennerhassett 1992)†. We conclude that the distortion has stabilized the basic flow, with the result that the Rayleigh modes start to decay. We also note that the growing velocity jump can be interpreted as a growing 'vortex sheet'. This is intriguing since the development of intense shear layers is one of the characteristic precursors to the formation of small-scale turbulence. This in turn raises the question of the stability of the thin shear layer to secondary disturbances.

Figure 8 displays results for $\theta = 60°$ and $\bar{\alpha} = 0.8$ on the right-hand branch of curve A. For this case $\bar{g}_r$ is again negative, and viscosity plays a similar role as in figure 6. It is worth noting that at moderate values of λ, viscosity can induce rather violent oscillations. However, the oscillations gradually disappear as λ increases, and ultimately the disturbance decays when λ is sufficiently large. The calculations presented here show that both λ and the sign of $\bar{g}_r$ determine the terminal form of the solution to (3.99). This conclusion is supported by other numerical calculations using artificial coefficients which we do not report here. However, it is worth observing that because $\bar{g}_r$ depends on θ, for any given wavenumber $\bar{\alpha}$ and sign of $\sum v_j \beta_j^{-\frac{4}{3}}$, it is *always* possible to find some θ such that $\bar{g}_r > 0$, i.e. such that a finite-time singularity can occur. In this sense, blow-up is more common than in the two-dimensional case (cf. Wu & Cowley 1993).

Of course, once the singularity occurs, our theory ceases to be valid. Nevertheless, the finite-time singularity indicates that an explosive growth is induced by nonlinear effects, and we suggest that this nonlinear blow-up may be the precursor to the bursting phenomena observed in experiments (e.g. Merkli & Thomann 1975; Hino *et al.* 1976). Moreover, as GC argue, the present theory does not break down until the amplitude of the disturbance becomes order one, at which point the flow will be governed by the Euler equations.

† For two-dimensional critical layers there can also be a linear growth in the mean flow; however at leading order this is a direct result of non-parallelism rather than nonlinear interactions within the critical layer (Goldstein & Hultgren 1988).

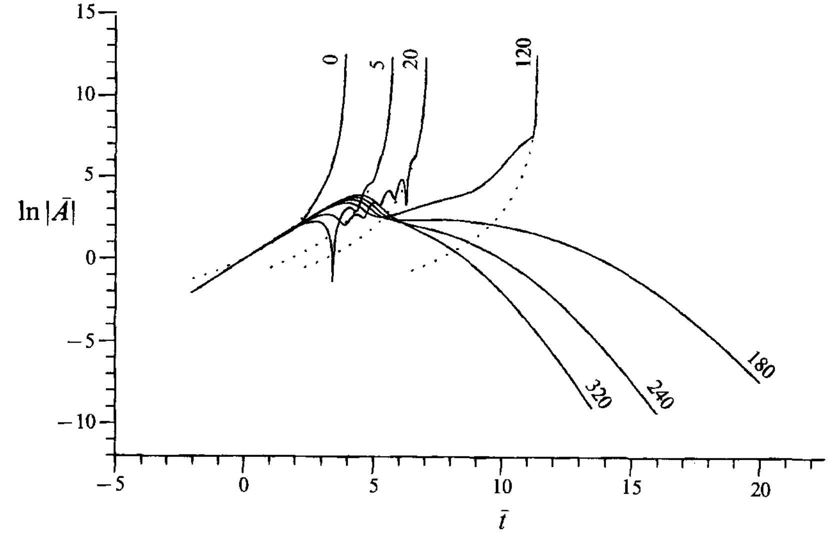

FIGURE 8. Ln$|\bar{A}|$ *vs.* the scaled time $\bar{t}$ for $\bar{\alpha} = 0.8$ and $\theta = 60°$. $\lambda = 0$, 5, 20, 120, 180, 240 and 320. Solid lines: numerical solutions; dotted lines: local asymptotic solutions (4.1).

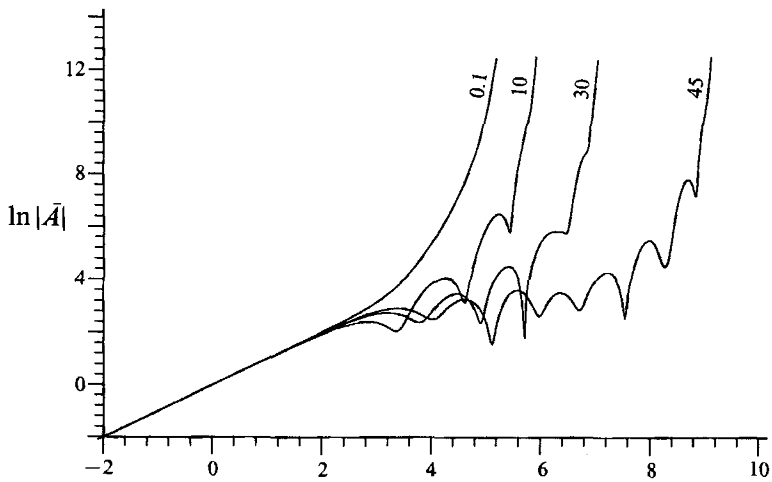

FIGURE 9. Ln$|\bar{A}|$ *vs.* the scaled time $\bar{t}$ for $\bar{\alpha} = 1.2$ and $\theta = 45°$. $\lambda = 0.1$, 10, 30, and 45.

Finally, we examine the special case $\theta = 45°$, again taking $\bar{\alpha} = 1.2$ on the right-hand branch of curve A as an example. This propagation angle is special because in *both* the inviscid ($\lambda = 0$) and viscous ($\lambda = +\infty$) limits, the nonlinear term in the amplitude equation vanishes. This does not occur, however, when $0 < \lambda < +\infty$. In figure 9, we depict evolution curves for four values of λ. The solutions clearly appear to develop a finite-time singularity. However, as λ is increased the solutions change significantly. As shown in figure 10, for sufficiently large λ, the solutions can evolve into a periodic oscillation without tending to a definite limit. In order to demonstrate that these solutions are not a numerical artifact, results are displayed at two different resolutions. We note that for $\lambda = 50.0$, the amplitude exhibits a rather 'chaotic'

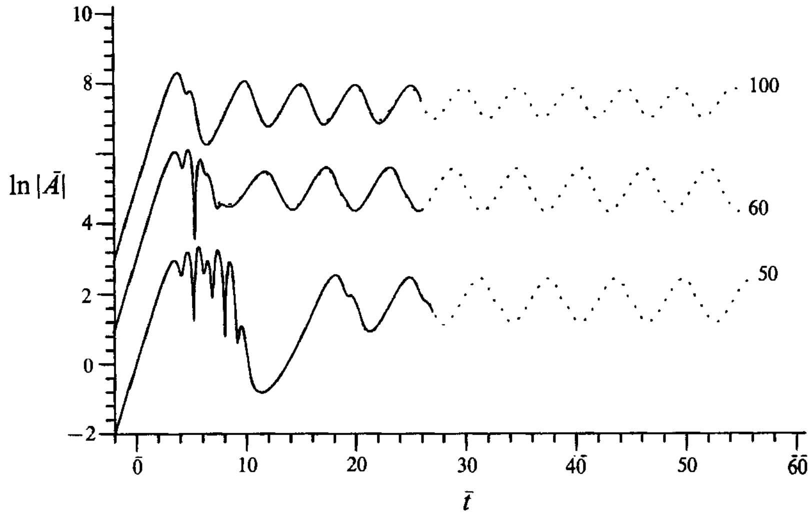

FIGURE 10. Ln$|\bar{A}|$ *vs.* the scaled time $\bar{t}$ for $\bar{\alpha} = 1.2$ and $\theta = 45°$. $\lambda = 50$, 60, and 100: to display the graph clearly, the curves for $\lambda = 60$ and 100 are shifted upwards by three and five units respectively. The integration time steps for $\lambda = 60$ and 100 are 0.05 (solid lines), and 0.1 (dotted lines). The integration time steps for $\lambda = 50$ are 0.025 (solid line) and 0.05 (dotted line).

transient state before relaxing into a periodic oscillation. This 'chaotic' transient becomes less noticeable, and ultimately disappears, as λ is increased further.

4.4. *The relationship with wave/vortex interactions*

A feature of critical-layer analyses is the fact that surprisingly large mean flows and vortices are generated, e.g. in our analysis a spanwise-dependent mean flow is generated which is as large as the fundamental disturbance. Similarly a feature of the wave/vortex interaction theory of Hall & Smith (1991) is that small-amplitude disturbances can modify the mean flow by an order-one amount. Further, both Rayleigh-wave/vortex interactions and our non-equilibrium critical-layer theory are based on almost-neutral three-dimensional instability waves and involve critical layers. It therefore seems natural to ask whether there is a link between the two theories. In particular, since the theory of Hall & Smith (1991) concerns the slow nonlinear evolution of weakly nonlinear travelling waves, could an initially linear disturbance consisting of a pair of oblique waves evolve through an 'unsteady' or 'non-equilibrium' critical-layer stage *en route* to a wave/vortex interaction? This is of course related to the questions:

(*a*) Can a wave/vortex interaction involving Rayleigh waves be established in the first place?

(*b*) Are the weakly nonlinear travelling waves in such a wave/vortex interaction stable?

Before partially addressing at least the first of these questions it is instructive to determine the range of validity of (4.7). From (1.3) and (4.6) it follows that the timescale over which the growth rate evolves, as specified by $\tilde{\tau}_1 = O(1)$, is

$$\tau - \tau_0 \sim O(\epsilon^{\frac{2}{3}} R^{\frac{1}{3}}) \; , \tag{4.16}$$

while from (1.1) and (B 7) the timescale over which the disturbance amplitude evolves, as specified by $\bar{t}_1 = O(1)$, is

$$\tau - \tau_0 \sim O(\epsilon^{-\frac{2}{3}} R^{-\frac{4}{3}}) . \tag{4.17}$$

Our theory assumes that these two timescales are distinct. This is no longer the case if

$$\epsilon \sim R^{-\frac{5}{4}} , \qquad \text{i.e.} \qquad \lambda \sim R^{\frac{1}{4}} , \tag{4.18}$$

since both timescales are then $O(R^{-\frac{1}{2}})$. We conclude that when

$$R^{-\frac{5}{4}} \ll \epsilon \ll R^{-1} , \qquad \text{or equivalently} \qquad 1 \ll \lambda \ll R^{\frac{1}{4}} , \tag{4.19}$$

the evolution can be described by (4.7). The validity of the *full* integro-differential form of the amplitude equation (3.87) is of course restricted only by $\lambda \ll R^{\frac{1}{4}}$.

From (4.6) the disturbance amplitude corresponding to the scaling (4.18) is $O(\lambda^{\frac{1}{3}}\epsilon)$, i.e. $O(R^{-\frac{7}{6}})$. In terms of the global Reynolds number $Re = R^2$, the disturbance amplitude is $O(Re^{-\frac{7}{12}})$, which we note is the (corrected) wave amplitude scale identified by Hall & Smith (1991) for Rayleigh-wave/vortex interactions. While a direct connection with that work is not possible, we note that as part of a study of weak Rayleigh-wave/vortex interactions, Brown, Brown & Smith (1993) have examined the scaling (4.18). Although they studied an equivalent spatial stability problem, in terms of our notation they effectively considered a problem with the three timescales, $t = R\tau$, $R^{\frac{1}{2}}\tau$ and τ. Their amplitude equation is similar to (4.7), but because the τ_1 and $\bar{t}_1$ timescales have merged, the coefficient of $\breve{A}$ in (4.7) becomes $g_0\bar{t}_1$. Also, as a result of a viscous sub-Stokes layer adjacent to the wall which makes an $O(R^{-\frac{1}{2}})$ correction to the growth rate (e.g. Cowley 1987), there is an additional linear term proportional to $\breve{A}$.

An important property of the amplitude equation derived by Brown *et al.* (1993) is that in one limit it can be reduced to a small-amplitude form of the wave/vortex interaction equations, while in another limit it reduces to (4.7), which itself is the 'highly viscous' limit of the full integro–differential equation (3.87). A mathematical link between the wave/vortex interaction equations, and the non-equilibrium critical-layer equations is thus made.

Let us now return to the discussion started in § 1 concerning a normal mode of amplitude ϵ_0 introduced at a time on one of the left-hand branches of the neutral curves. We note:

(*a*) if $\epsilon_0 = O(R^{-\frac{7}{6}})$, then the evolution of the disturbance is described by the nonlinear equation derived by Brown *et al.* (1993);

(*b*) if $\epsilon_0 = o(R^{-\frac{7}{6}})$, but $\log \epsilon_0^{-1} \ll R$, the nonlinear evolution of the disturbance is described by (3.87), or its limiting forms (4.7) and (3.92).

In the second case we have seen either that the disturbance hits a finite-time singularity, or that the mean flow is stabilized with the result that the disturbance decays exponentially. In both cases the final behaviour is described by much more rapid timescales than would be necessary for the flow to evolve to a slowly varying wave/vortex interaction. Thus at least for a disturbance consisting of a pair of equal-amplitude oblique waves, it seems that for a broad range of initial amplitudes the flow does not develop into the wave/vortex interaction regime of Hall & Smith (1991). However, note that when exponential decay sets in near a left-hand neutral curve, we anticipate that the disturbance will start to grow again over the slow timescale τ_1 as a result of an increase in the linear coefficient of (4.7). The subsequent nonlinear evolution of such a disturbance will depend, *inter alia*, on the form of the critical layer when the flow becomes nonlinear again.

5 Discussion and conclusions

We have derived an integro-differential amplitude equation, i.e. (3.99), that describes the evolution of a pair of oblique waves in inviscidly unstable shear layers such as Stokes layers. We have assumed that nonlinear effects become important while the local growth rate of the disturbances is small (although not too small), and have extended the previous analysis of GC by including viscosity. We have also relaxed the condition that the critical layer(s) occur at inflexion points, although our results remain valid if they do (e.g. as is the case for free shear layers). However, we require that the spanwise wavenumber β is not too small so that the nonlinearity associated with the logarithmic branch point singularity is much weaker than that associated with the pole. A scaling argument, similar to that in Wu & Cowley (1993), shows that this condition is violated when

$$\beta \sim O(\epsilon^{\frac{1}{5}}) ,$$

because the nonlinearity from the simple-pole singularity is then as strong as that from the branch-point singularity. This problem has been considered by Wu (1993*a*) who allows the disturbance amplitude to be modulated both in the spanwise direction on a scale $Z = \epsilon^{\frac{1}{5}} z$, and in time on the scale $t_1 = \epsilon^{\frac{2}{5}} t$.

Numerical solutions of the amplitude equation (3.99) either blow-up in a finite-time singularity, or decay exponentially at large times. Similar behaviour is found analytically for the reduced equation (4.7) that describes the amplitude evolution in the very viscous limit. Equilibrium solutions to the amplitude equations could not be found, which suggests that nonlinear effects do not lead the disturbance to saturate.

A direct comparison with experiment is hard, since we have been unable to find experiments on Stokes layers studying the evolution of well-controlled initial disturbances. Indeed, in most experiments the instabilities are allowed to develop from background noise. However, in an attempt to relate our theory to such experiments, let us suppose that instability modes are excited continuously. Then, because there are well-defined left-hand branches to the neutral curves A and B in figure 1, there are specific times when new modes with a particular wavenumber can be excited. Let us assume that these modes are excited as soon as they are 'viable'.

If the initial amplitudes of these disturbances are *extremely* small, then their evolution as the Stokes layer slowly changes can be fully described by linear theory — eventually the disturbances either equilibrate at a finite amplitude or they decay on reaching the right-hand branch of a neutral curve (Cowley 1987). However, if their initial amplitude is slightly larger, then the evolution of the disturbances can become nonlinear near the right-hand branch of the neutral curve. Since the most rapidly growing linear modes are two-dimensional, then on the basis of linear theory, such disturbances are likely to have the largest amplitudes. This two-dimensional case has been considered by Wu & Cowley (1993) who show that for disturbances with certain wavelengths a finite time-singularity can develop no matter how large viscous effects are. However, of possibly greater importance to experimentally observed transition is the fact that an inviscid disturbance consisting of a resonant-triad of waves can be preferentially amplified by means of parametric resonance so that small-amplitude oblique modes can attain large amplitudes through a period of super-exponential growth (Goldstein & Lee 1992; Wu 1992). Further, Goldstein & Lee (1992) and

Wu (1992) have shown that once the oblique waves are sufficiently large, a nonlinear backreaction causes such a resonant-triad of waves to always develop a finite-time singularity — as hypothesized above this may lead to transition to turbulence (see also Goldstein & Lee 1992). Wu's (1992) analysis was inviscid, and it seems hard to assess analytically the influence of viscosity on this conclusion, because a study of the viscous resonant triad would involve extremely complicated algebra. Rather than take such an approach here, we have supposed that there is a preferential mechanism for exciting three-dimensional disturbances, e.g. small grooves in the plate. Of course, it may turn out that our analysis is directly relevant to the viscous resonant-triad case if oblique mode interactions become dominant in one of the later stages of evolution.† Whatever the method of excitation our results show that for moderate-sized disturbances (*a*) for a range of 'obliqueness' angles the disturbances decay as a result of viscous effects, but (*b*) for other angles a finite-time singularity arises accompanied by an explosive increase in wave-amplitude. For slightly larger disturbances viscous effects have no time to act, and for all obliqueness angles there is an explosive increase in wave amplitude.

At even larger levels of background noise a disturbance of given wavelength will become nonlinear at times well before the right-hand branch of the neutral curve, and our theory is not applicable. A further increase in the background disturbance level means that nonlinear effects need to be included at times near the left-hand branch of the neutral curve. Again, two-dimensional disturbances have the most rapid linear growth. However, over much of the left-hand branch of curve A, the two-dimensional critical layer is regular, and so nonlinear effects lead to algebraic rather than exponential growth (e.g. Huerre & Scott 1980; Churilov & Shukhman 1987*a*; Goldstein & Hultgren 1988). Therefore, three-dimensional instabilities are potentially important, although a full analysis would involve testing the stability of the flow with a two-dimensional, quasi-equilibrium, critical layer to secondary three-dimensional disturbances. This is an extensive calculation, which we do not tackle here. Moreover, as Kelly & Maslowe (1970), Killworth & McIntyre (1985) and Haynes (1985) have observed, the quasi-equilibrium critical layer itself may be unstable to very high-frequency *two*-dimensional disturbances with wavelengths comparable to the thickness of the critical layer.

Instead we note that when three-dimensional disturbances are preferentially excited so that their evolution becomes *inviscidly* nonlinear near a left-hand branch, explosive growth of the amplitude will again occur for all pairs of oblique modes. However, if the *initial* disturbance is sufficiently *large*, then viscous effects mean that explosive growth can only occur for a range of obliqueness angles. We believe that this is experimentally significant because it implies that for a relatively large range of initial disturbances, i.e. $\epsilon_0 = o(R^{-\frac{7}{6}})$ but $\log \epsilon_0^{-1} \ll R^{\frac{1}{3}}$, there is a spread of obliqueness angles for which viscous effects cannot prevent explosive growth. This is so whether or not viscous effects can force two-dimensional disturbances to evolve into quasi-equilibrium states. For even larger initial disturbances, i.e. $\epsilon_0 = O(R^{-\frac{7}{6}})$, the amplitude equation is modified to that of Brown *et al.* (1993).

Our results suggest that experimental observations are likely to depend on the background level of disturbances. In particular, the lower the background level of

† In the later stages of the inviscid resonant triad it is the self-interaction term in the oblique mode equation which causes the singularity. The rapid growth of the oblique mode is then transferred to the two-dimensional mode through the backreaction terms in the two-dimensional equation. This results in a singularity in the two-dimensional mode.

disturbance, the later in the oscillation cycle that nonlinear disturbances should be observed. For example, Monkewitz & Bunster (1987) note that 'the first visible finite amplitude disturbances appear shortly before and around flow reversal at the edge of the boundary layer'; our theory when applied close to the right-hand branch of curve A should be relevant in this case. On the other hand, Akhavan, Kamm & Shapiro (1991*a*, *b*) note that at relatively large Reynolds numbers 'turbulence appeared explosively towards the end of the acceleration phase'. This is a little earlier than could directly be explained using our theory applied to the left-hand branch of curve A, but we note that Akhavan *et al.* (1991*a*, *b*) conducted their experiments on finite-width Stokes layers in a pipe, and that their ensemble-averaged velocity profiles differ from the laminar profiles that we have assumed.

As in GC, an important feature of our results is that nonlinear interactions inside the critical layers generate in the main part of the flow both a spanwise-dependent mean flow of the same size as the fundamental wave, and a longitudinal vortex. We note that a relatively strong spanwise-dependent mean flow was observed by Hino *et al.* (1983) in a finite Stokes layer between two plates. Hino *et al.* (1983) also suggested that the vortex structures and the bursting processes that they observed in Stokes layers are similar to those observed in turbulent boundary layers. This is possibly not surprising given (*a*) that our theory is applicable to any high-Reynolds-number shear flow that supports Rayleigh waves, and (*b*) that the large-scale coherent structures in the outer region of a turbulent boundary layer may generate such shear flows in the wall region. Thus the results obtained here may be applicable to the understanding of wall-layer phenomena in turbulent boundary layers such as 'streaky' vortices and high-frequency bursts.

In addition we recall that the formation of Λ-vortices in boundary-layer transition has been linked with the secondary instability of Tollmien–Schlichting waves and longitudinal vortices to high-frequency (Rayleigh) modes, e.g. Betchov (1960), Greenspan & Benney (1963). Since the Tollmien–Schlichting waves and longitudinal vortices are quasi-two-dimensional in the sense that their wavelengths are much larger than the thickness of the boundary layer, a slight modification to our analysis should yield an alternative nonlinear approach to describing the 'spike' stage of transition. Moreover, the generalization of the analysis to three-dimensional basic states should also make it possible to extend Hall & Horseman's (1991) linear secondary instability analysis of Görtler vortices into the nonlinear regime. We also note that it is straightforward to extend our analysis to compressible flow. In particular, essentially the same amplitude evolution equation, i.e. (3.99), is obtained since the perturbation velocities in the critical layer are subsonic. Our results are especially applicable to supersonic compressible flows since the most rapidly growing linear disturbances are then three-dimensional. Moreover, we note that the same amplitude equation applies to the interaction of a pair of helical modes spinning in opposite directions in axisymmetric shear layers, e.g. axisymmetric jets. The reason is that the critical layer there, though annular, is locally 'flat' in the sense that the variation of the flow is much more rapid in the radial direction than in the circumferential direction.

Another consequence of our work is its relationship to the Rayleigh-wave/vortex interaction of Hall & Smith (1991). An implicit assumption in the Hall–Smith theory is that the weakly nonlinear Rayleigh wave, whose slow evolution forces the order-one change in the mean flow, is stable. At first sight this assumption seems reasonable given, for instance, that linearly unstable Görtler vortices can evolve through a weakly nonlinear stage to a strongly nonlinear stage in which the mean flow is distorted by an order-one amount (Hall 1991).

Initially we anticipated that our linear disturbances would first evolve through the weakly nonlinear stage described in this paper and then develop into a wave/vortex interaction at later times (e.g. as a result of saturating into an equilibrium state). However, we found that the disturbances either evolve to an 'Euler' stage through a finite time singularity (GC), or they decay exponentially. Similar results have been obtained independently by Brown *et al.* (1993) for disturbances which become nonlinear in the $|\Delta\tau| = O(R^{-\frac{1}{3}})$ asymptotic regime near a left-hand neutral curve.

We conclude that for a wide range of initial amplitudes, a pair of oblique modes do not evolve to a Rayleigh-wave/vortex interaction, even though such a disturbance seems a natural initial condition for such flows. It remains to be checked that this is still the case for a pair of oblique waves of *unequal* amplitude. In addition, the effect of both streamwise and spanwise *modulation* of the waves should be investigated, especially on the form of the finite-time singularity.

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Appendix A

In this appendix we combine our results for the Stokes layer with those of GC for a shear layer, to deduce the viscous version of the amplitude evolution equation for a shear layer.

First, we introduce a viscous parameter λ by scaling the local Reynolds number $R = \delta_0\Delta/\nu$ in the following way:

$$R^{-1} = \lambda\epsilon \; . \tag{A 1}$$

Then the amplitude equation is

$$\frac{1}{\bar{\kappa}}\frac{\mathrm{d}A}{\mathrm{d}\bar{x}} = A - \tfrac{1}{2}\gamma\tan^2\theta\int_0^{+\infty}\int_0^{+\infty}\tilde{K}(\xi,\eta|\lambda)A(\bar{x}-\xi)A(\bar{x}-\xi-\eta)A^*(\bar{x}-2\xi-\eta)\mathrm{d}\xi\mathrm{d}\eta \; , \tag{A 2}$$

where the kernel $\tilde{K}$ is defined by (3.85) — the suffix j has been dropped because there is only one critical layer in a free shear layer. The parameter s_1 involved in $\tilde{K}$ now should be defined by

$$s_1 = \tfrac{1}{3}(\alpha U_c')^2\lambda/[-\tfrac{1}{2}S_1U_cU_c']^3 \; . \tag{A 3}$$

Readers should consult GC for the definitions of δ, S_1, U_c etc. The constants $\bar{\kappa}$ and γ are defined by equations (3.70) and (3.71) of GC respectively.

Appendix B

In this appendix, we show that the integro-differential amplitude equation (3.87) reduces to (4.4) as $\lambda \to +\infty$.

In order to obtain an asymptotic estimate of the nonlinear term, we first split it into the following sum:

$$\begin{aligned} N &\equiv \int_0^{+\infty}\int_0^{+\infty} K_j(\xi,\eta|\lambda)A(t_1-\xi)A(t_1-\xi-\eta)A^*(t_1-2\xi-\eta)\mathrm{d}\xi\mathrm{d}\eta \\ &= N^{(0)} + N^{(1)} + 2\sin^2\theta[N^{(2)} + N^{(3)}] + 8\sin^4\theta N^{(4)} + N^{(5)} \; , \end{aligned} \tag{B 1}$$

where

$$N^{(0)} = 2\int_0^{+\infty}\int_0^{+\infty} \tilde{K}^{(0)}(\xi,\eta)\xi^3 A(t_1-\xi)A(t_1-\xi-\eta)A^*(t_1-2\xi-\eta)\mathrm{d}\xi\mathrm{d}\eta\ , \qquad \text{(B 2)}$$

$$N^{(1)} = \int_0^{+\infty}\int_0^{+\infty} \tilde{K}^{(0)}(\xi,\eta)\xi^2\eta A(t_1-\xi)A(t_1-\xi-\eta)A^*(t_1-2\xi-\eta)\mathrm{d}\xi\mathrm{d}\eta\ , \qquad \text{(B 3)}$$

$$N^{(2)} = \int_0^{+\infty}\int_0^{+\infty}\left\{\tilde{K}^{(0)}(\xi,\eta)\int_0^{\eta}[\xi^2+2\xi(\eta-\zeta)]\mathrm{e}^{-2s_1\zeta^3-3s_1\xi\zeta^2}\mathrm{d}\zeta\right\} \times A(t_1-\xi)A(t_1-\xi-\eta)A^*(t_1-2\xi-\eta)\mathrm{d}\xi\mathrm{d}\eta\ , \qquad \text{(B 4)}$$

$$N^{(3)} = \int_0^{+\infty}\int_0^{+\infty}\left\{\tilde{K}^{(0)}(\xi,\eta)\int_0^{\xi}[\zeta(2\eta+3\zeta)-\xi(\xi+2\eta+2\zeta)]\mathrm{e}^{-3s_1\xi\zeta^2}\mathrm{d}\zeta\right\} \times A(t_1-\xi)A(t_1-\xi-\eta)A^*(t_1-2\xi-\eta)\mathrm{d}\xi\mathrm{d}\eta\ , \qquad \text{(B 5)}$$

$$N^{(4)} = \int_0^{+\infty}\int_0^{+\infty} \left\{\tilde{K}^{(0)}(\xi,\eta)\int_0^{\xi}\mathrm{d}\zeta\,\mathrm{e}^{-3s_1\xi\zeta^2}\int_0^{\eta+\zeta}(\upsilon-\eta-\zeta)[1+6s_1(\xi-\zeta)\zeta^2]\mathrm{e}^{-s_1(2\upsilon^3+3\xi\upsilon^2)}\mathrm{d}\upsilon\right\} \times A(t_1-\xi)A(t_1-\xi-\eta)A^*(t_1-2\xi-\eta)\mathrm{d}\xi\mathrm{d}\eta\ , \qquad \text{(B 6)}$$

and $N^{(5)}$ denotes the rest of the nonlinear term, i.e. the part associated with the fourth, fifth, seventh and eighth terms of the right-hand side of (3.85).

In order to estimate these integrals as $\lambda \to \infty$, it is necessary to make one of the following changes of variables:

substitution I:

$$t_1 = \lambda^{\frac{1}{3}}\bar{t}_1\ , \quad \zeta = \lambda^{-\frac{1}{3}}\bar{\zeta}\ , \quad \upsilon = \lambda^{-\frac{1}{3}}\bar{\upsilon}\ , \qquad \text{(B 7)}$$

$$\xi = \lambda^{-\frac{2}{3}}\bar{\xi}\ , \quad \eta = \lambda^{\frac{1}{3}}\bar{\eta}\ ; \qquad \text{(B 8)}$$

substitution II: (B 7) and

$$\xi = \lambda^{-\frac{1}{3}}\tilde{\xi}\ , \quad \eta = \lambda^{-\frac{1}{3}}\tilde{\eta}\ . \qquad \text{(B 9)}$$

The appropriate substitution must be chosen in order that the resulting integrals are convergent. We find that substitution II converts the nonlinear terms into a classical cubic form, i.e. the history effects are damped out.

Applying substitution II to $N^{(0)}$ and $N^{(5)}$, and taking the limit $\lambda \to +\infty$, we have

$$N^{(0)} \to \lambda^{-\frac{5}{3}}B_0 A|A|^2\ , \qquad \text{(B 10)}$$

and

$$N^{(5)} \to \lambda^{-\frac{5}{3}}C_0 A|A|^2\ , \qquad \text{(B 11)}$$

where B_0 and C_0 are constants defined by convergent integrals. However if we perform substitution II in $N^{(1)}$, the resultant integral diverges. Instead, we integrate $N^{(1)}$ by parts with respect to η, and write it in the form:

$$N^{(1)} = (-3s_1)^{-1}\int_0^{+\infty}\int_0^{+\infty}\tilde{K}^{(0)}(\xi,\eta)\eta A(t_1-\xi)\frac{\partial}{\partial t_1}[A(t_1-\xi-\eta)A^*(t_1-2\xi-\eta)]\mathrm{d}\xi\mathrm{d}\eta$$
$$-(3s_1)^{-2}\int_0^{+\infty}\int_0^{+\infty}\xi^{-2}[\mathrm{e}^{-3s_1\xi^2\eta}-1]\mathrm{e}^{-2s_1\xi^3}A(t_1-\xi)\frac{\partial}{\partial t_1}[A(t_1-\xi-\eta)A^*(t_1-2\xi-\eta)]\mathrm{d}\xi\mathrm{d}\eta\ . \qquad \text{(B 12)}$$

We now perform substitution I and take the limit $\lambda \to +\infty$; we obtain

$$N^{(1)} \to \frac{\sqrt{3}}{18}\Gamma(\frac{1}{2})\lambda^{-\frac{4}{3}}(\beta_j)^{-\frac{3}{2}}A\frac{\partial}{\partial \bar{t}_1}\int_0^{+\infty}\bar{\eta}^{\frac{1}{2}}|A(\bar{t}_1-\bar{\eta})|^2\mathrm{d}\bar{\eta}\ , \tag{B 13}$$

where $\beta_j = \frac{1}{3}\alpha^2\bar{U}_y^2(y_c^j)$. Note that substitution I leaves the integral convergent by virtue of the exponential decay of A as $\bar{\eta} \to +\infty$.

In the case of $N^{(2)}$, we first make the transformation $(\eta - \zeta) \to \eta$ to obtain

$$\begin{aligned} N^{(2)} = &\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \xi^2 \mathrm{e}^{-2s_1\xi^3-3s_1\xi^2(\eta+\zeta)-2s_1\zeta^3-3s_1\xi\zeta^2} \\ &\qquad \times A(t_1-\xi)A(t_1-\xi-\eta-\zeta)A^*(t_1-2\xi-\eta-\zeta)\mathrm{d}\zeta\mathrm{d}\xi\mathrm{d}\eta \\ &+\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}[2\xi\eta]\mathrm{e}^{-2s_1\xi^3-3s_1\xi^2(\eta+\zeta)-2s_1\zeta^3-3s_1\xi\zeta^2} \\ &\qquad \times A(t_1-\xi)A(t_1-\xi-\eta-\zeta)A^*(t_1-2\xi-\eta-\zeta)\mathrm{d}\zeta\mathrm{d}\xi\mathrm{d}\eta\ . \end{aligned} \tag{B 14}$$

Performing substitutions II and I into the first and second integrals respectively, we find that the first term is order $\lambda^{-\frac{5}{3}}$, while the second term is order λ^{-1}. More precisely,

$$N^{(2)} = \lambda^{-1}\beta_j^{-\frac{4}{3}}\frac{1}{9\times 2^{\frac{1}{3}}}\Gamma(\tfrac{1}{3})A\int_0^{+\infty}|A(\bar{t}_1-\bar{\eta})|^2\mathrm{d}\bar{\eta} + O(\lambda^{-\frac{4}{3}})\ . \tag{B 15}$$

In order to estimate $N^{(3)}$ we first integrate by parts to obtain

$$\begin{aligned} N^{(3)} = &-\int_0^{+\infty}\int_0^{+\infty}\mathrm{e}^{-5s_1\xi^3-3s_1\xi^2\eta}\xi^3A(t_1-\xi)A(t_1-\xi-\eta)A^*(t_1-2\xi-\eta)\mathrm{d}\xi\mathrm{d}\eta \\ &-\int_0^{+\infty}\int_0^{+\infty}\mathrm{e}^{-5s_1\xi^3-3s_1\xi^2\eta}\xi^2\eta A(t_1-\xi)A(t_1-\xi-\eta)A^*(t_1-2\xi-\eta)\mathrm{d}\xi\mathrm{d}\eta \\ &+\int_0^{+\infty}\int_0^{+\infty}\tilde{K}^{(0)}(\xi,\eta)\int_0^{\xi}(6s_1\xi\zeta)[\eta\zeta^2+\zeta^3-\xi^2\zeta-2\xi\eta\zeta-\xi\zeta^2]\mathrm{e}^{-3s_1\xi\zeta^2} \\ &\qquad \times A(t_1-\xi)A(t_1-\xi-\eta)A^*(t_1-2\xi-\eta)\mathrm{d}\zeta\mathrm{d}\xi\mathrm{d}\eta\ . \end{aligned} \tag{B 16}$$

Performing substitution II in the first and third terms, we can show that these two terms tend to $\lambda^{-\frac{5}{3}}D_0A|A|^2$, where D_0 is a constant defined by a convergent integral. The second term is similar to $N^{(1)}$; the only difference is that the exponent $5s_1\xi^3$ now replaces $2s_1\xi^3$ of $N^{(1)}$ (see (3.68) for the definition of $\tilde{K}^{(0)}$). We conclude that as $\lambda \to +\infty$,

$$N^{(3)} \sim O(\lambda^{-\frac{4}{3}})\ . \tag{B 17}$$

Finally for $N^{(4)}$ we first perform the transformation $(\eta - \upsilon) \to \eta$, and then we write it in three parts:

$$\begin{aligned} N^{(4)} = &\int_0^{+\infty}\mathrm{d}\xi\int_0^{\xi}\mathrm{d}\zeta\mathrm{e}^{-3s_1\xi\zeta^2}\int_0^{\zeta}\mathrm{d}\eta\int_{\eta}^{+\infty}\mathrm{e}^{-2s_1\xi^3-3s_1\xi^2(\upsilon-\eta)-s_1(2\upsilon^3+3\xi\upsilon^2)} \\ &\qquad \times(\eta-\zeta)[1+6s_1(\xi-\zeta)\zeta^2]A(t_1-\xi)A(t_1-\xi+\eta-\upsilon)A^*(t_1-2\xi+\eta-\upsilon)\mathrm{d}\upsilon \\ &-\int_0^{+\infty}\mathrm{d}\xi\int_0^{+\infty}\mathrm{d}\eta\int_0^{\xi}\mathrm{d}\zeta\mathrm{e}^{-3s_1\xi\zeta^2}\int_0^{+\infty}\mathrm{e}^{-2s_1\xi^3-3s_1\xi^2(\eta+\upsilon)-s_1(2\upsilon^3+3\xi\upsilon^2)} \\ &\qquad \times[\zeta+6s_1(\xi-\zeta)\zeta^2(\eta+\zeta)]A(t_1-\xi)A(t_1-\xi-\eta-\upsilon)A^*(t_1-2\xi-\eta-\upsilon)\mathrm{d}\upsilon \\ &-\int_0^{+\infty}\mathrm{d}\xi\int_0^{+\infty}\mathrm{d}\eta\int_0^{\xi}d\zeta\mathrm{e}^{-3s_1\xi\zeta^2}\int_0^{+\infty}\mathrm{e}^{-2s_1\xi^3-3s_1\xi^2(\eta+\upsilon)-s_1(2\upsilon^3+3\xi\upsilon^2)} \\ &\qquad \times\eta A(t_1-\xi)A(t_1-\xi-\eta-\upsilon)A^*(t_1-2\xi-\eta-\upsilon)\mathrm{d}\upsilon\ . \end{aligned} \tag{B 18}$$

By use of substitution II we can show that the first and the second terms tend to $(\lambda^{-\frac{5}{3}}E_0A|A|^2)$ as $\lambda \to +\infty$; here E_0 is again a constant defined by a convergent integral. Use of substitution I in the third term shows that it contributes a leading-order term of order λ^{-1}:

$$N^{(4)} = -\lambda^{-1}\beta_j^{-\frac{4}{3}}\frac{1}{18\times 2^{\frac{1}{3}}}\Gamma(\tfrac{1}{3})A\int_0^{+\infty}|A(\bar{t}_1-\bar{\eta})|^2\mathrm{d}\bar{\eta} + O(\lambda^{-\frac{4}{3}})\ . \tag{B 19}$$

Combining (B 15) and (B 19), and using (B 1), (B 10), (B 11), (B 13) and (B 17), we obtain the final estimate for N:

$$N \to \lambda^{-1}\beta_j^{-\frac{4}{3}}\frac{2^{\frac{2}{3}}}{9}\sin^2\theta[1-2\sin^2\theta]\Gamma(\tfrac{1}{3})A\int_0^{+\infty}|A(\bar{t}_1-\bar{\eta})|^2\mathrm{d}\bar{\eta} + O(\lambda^{-\frac{4}{3}})\ . \tag{B 20}$$

Using (B 20), (3.87) and (3.15) we obtain (4.4).

Appendix C

As $\bar{t} \to -\infty$, the amplitude equation (3.99) has a solution of the following form:

$$\bar{A} = \sum_{n=0}^{+\infty} a_k \mathrm{e}^{(2n+1)\bar{t}}\ . \tag{C 1}$$

Substituting the above expression into (3.99), equating the coefficients of $\mathrm{e}^{(2n+1)\bar{t}}$, and setting $a_0 = 1$, we obtain the recursion relation

$$a_{n+1} = \frac{1}{2(n+1)}\sum_{l=0}^{n}\sum_{k=0}^{l} a^*_{n-l}a_k a_{l-k}\sum_j \nu_j Q_j(k,l,n)\ , \qquad n = 0,1,2,\dots\ , \tag{C 2}$$

where

$$Q_j(k,l,n) = \int_0^{+\infty}\int_0^{+\infty} K_j(\xi,\eta|\lambda)\mathrm{e}^{-2(2n-l+2)\xi-2(n+k-l+1)\eta}\mathrm{d}\xi\mathrm{d}\eta\ . \tag{C 3}$$

For simplicity, we only consider the inviscid limit $\lambda = 0$. In this case, we can integrate $Q(k,l,n)$ analytically to yield

$$Q_j(k,l,n) = \frac{1}{16}\left\{\frac{6(1-2\sin^2\theta)}{(n+k-l+1)(2n-l+2)^4} + \frac{1-4\sin^4\theta}{(n+k-l+1)^2(2n-l+2)^3} + \frac{2\sin^2\theta(1-2\sin^2\theta)}{(n+k-l)^3(2n-l+2)^2}\right\}\ . \tag{C 4}$$

Appendix D

In this appendix, we show that although the wave decays exponentially, the nonlinearly induced streamwise slip velocity (3.49) actually grows linearly with the time t_1. Without loss of generality, we consider

$$\Delta U = \int_0^{+\infty}\int_0^{\eta}(\eta-\zeta)\mathrm{e}^{-2s_1\zeta^3}|A(t_1-\eta)|^2\mathrm{d}\zeta\mathrm{d}\eta\ . \tag{D 1}$$

This is essentially the shear produced by nonlinear interactions inside the critical layer. By means of the transform $(t_1-\eta) \to \eta$, we can write ΔU as the sum of three terms

$$\Delta U = t_1 A_1 - A_2 - A_3 , \tag{D 2}$$

where

$$A_1(t_1) = \int_{-\infty}^{t_1} |A(\eta)|^2 d\eta \int_0^{t_1-\eta} e^{-2s_1\zeta^3} d\zeta , \tag{D 3}$$

$$A_2(t_1) = \int_{-\infty}^{t_1} |A(\eta)|^2 d\eta \int_0^{t_1-\eta} \zeta e^{-2s_1\zeta^3} d\zeta , \tag{D 4}$$

$$A_3(t_1) = \int_{-\infty}^{t_1} \eta |A(\eta)|^2 d\eta \int_0^{t_1-\eta} e^{-2s_1\zeta^3} d\zeta . \tag{D 5}$$

Since the integrands in (D 3) are positive, we have that

$$A_1 \leq \int_{-\infty}^{t_1} |A(\eta)|^2 d\eta \int_0^{+\infty} e^{-2s_1\zeta^3} d\zeta \leq \int_{-\infty}^{+\infty} |A(\eta)|^2 d\eta \int_0^{+\infty} e^{-2s_1\zeta^3} d\zeta . \tag{D 6}$$

Similarly we find that

$$A_2 \leq \int_{-\infty}^{+\infty} |A(\eta)|^2 d\eta \int_0^{+\infty} \zeta e^{-2s_1\zeta^3} d\zeta . \tag{D 7}$$

and

$$|A_3| \leq \int_{-\infty}^{t_1} |\eta A(\eta)|^2 d\eta \int_0^{t_1-\eta} e^{-2s_1\zeta^3} d\zeta \leq \int_{-\infty}^{+\infty} |\eta A(\eta)|^2 d\eta \int_0^{+\infty} e^{-2s_1\zeta^3} d\zeta . \tag{D 8}$$

The estimates (D 6)–(D 8) mean that only the term involving A_1 contributes to the leading-order term in (D 1). Moreover, $A_1(t_1)$ is a monotonically increasing function. Hence after a little algebra we conclude from (D 6) that as $t_1 \to +\infty$,

$$A_1(t_1) \to \int_{-\infty}^{+\infty} |A(\eta)|^2 d\eta \int_0^{+\infty} e^{-2s_1\zeta^3} d\zeta > 0 , \tag{D 9}$$

and that

$$\Delta U \to \kappa t_1 , \tag{D 10}$$

where $\kappa = A_1(+\infty)$.

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